MAT3280 Midterm exam – sample solution

10/28/2023

Question 1. (10 points) Suppose we have a mechanism to generate a random variable U that is uniformly distributed between 0 and 1. Find a function h such that $X = h(U)$ is exponentially distributed with mean $1/\lambda$, where λ is a fixed positive constant.

The probability density function of exponential random variable is

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x \geq 0, \\ 0 & \text{for } x < 0. \end{cases}$$

Solution. The cumulative distribution function of an exponentially distributed random variable is

$$F_X(x) = \begin{cases} 1 - e^{-\lambda x} & \text{for } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

We solve for the inverse function of $F_X(x)$ by solving

$$y = F_X(x).$$

The inverse function is

$$h(y) = -\frac{1}{\lambda} \ln(1 - y),$$

where y is a variable between 0 and 1. We can take the function h and define random variable X by

$$X = -\frac{1}{\lambda} \ln(1 - U)$$

The distribution of X will be exponential with mean $1/\lambda$. To see why X has the desired distribution, we can take any positive real number x, and consider the probability

$$\begin{aligned} \Pr(X \leq x) &= \Pr\Big(-\frac{1}{\lambda} \ln(1 - U) \leq x \Big) \\ &= \Pr(U \leq 1 - e^{-\lambda x}). \end{aligned}$$

Because U is uniformly distributed, the last probability is equal to $1 - e^{-\lambda x}$. Therefore

$$\Pr(X \leq x) = 1 - e^{-\lambda x}.$$

This proves that the cdf of X is the same as the cdf of the exponential distribution with mean $1/\lambda$.

(Alternate solution: Since U and $1-U$ has the same distribution, we can simplify a little bit and take $h_1(u) = -\lambda^{-1}\ln(u)$. The random variable $h_1(U)$ has the same distribution as $h(U)$.)

Question 2. (10 points) Let $G(x)$ be the function

$$G(x) \triangleq \begin{cases} 1 - \frac{1}{x^2+2} & \text{for } x > 0, \\ 0 & \text{for } x \leq 0. \end{cases}$$

Is it possible to construct a real-valued random variable X such that the probability $P(X \leq x)$ is equal to $G(x)$, for all real numbers x?

Solution. The function $G(x)$ is not a Stieltjes measure function, because at the point $x = 0$, the function $G(x)$ is continuous from the left, not continuous from the right. Based on this observation, the answer to this question is no.

In more detail, suppose such a random variable X exists. This means that there is a probability space $(\Omega, \mathscr{F}, P)$, and $X(\omega)$ is a measurable function such that for all real numbers x, the event

$$\{\omega \in \Omega : \ X(\omega) \leq x\}$$

has probability $G(x)$. We would have $P(X \leq 0) = G(0) = 0$. However, if we consider the events

$$A_n \triangleq \{\omega \in \Omega : X(\omega) \leq 1/n\}$$

for positive integer n, we see that the events A_n form a decreasing sequence of sets, with intersection

$$B \triangleq \{\omega \in \Omega : \ X(\omega) \leq 0\}$$

By upper semi-continuity of measure, which should hold in general, we would get

$$\lim_{n\to\infty} P(A_n) = P(B) = G(0) = 0.$$

However, the numerical value of the limit on the left is

$$\lim_{n\to\infty} P(A_n) = \lim_{n\to\infty} 1 - \frac{1}{(1/n)^2 + 2} = \frac{1}{2}.$$

This is a contradiction and hence no such random variable X exists.

Question 3. (16 points) Let $\mathscr{F}$ denote the σ-algebra generated by closed intervals of the form $[a, b]$, where $a \leq b$ are rational numbers. Show that the following sets are $\mathscr{F}$-measurable.

(a) The set of all integers.

(b) Open interval (x_1, x_2), where x_1 and x_2 are real numbers with $x_1 < x_2$.

Solution.

(a) If we set $a = b$, the closed interval $[a, b]$ reduces to a single point. By taking a and b to be equal to an integer n, which is certainly a rational number, we see that the set $\{n\}$ consisting of

integer n is $\mathscr{F}$-measurable. Since the set of integers are countable, we can write it as a countable union

$$\mathbb{Z} = \bigcup_{n\in\mathbb{Z}} \{n\}.$$

Since $\mathscr{F}$ is supposed to be closed under countable untion, we conclude that the set of all integers is $\mathscr{F}$-measurable.

(b) We take a sequence of rational numbers a_n, that are all strictly larger than x_1 and converging to x_1 from the right, and take another sequence of rational numbers b_n, that are all strictly less than x_2 and converging to x_2 from the left.

The reason why we can pick these two sequences is because rational numbers are dense in the real number line. We can start with choosing a_1 to be any rational number that is strictly larger than x_1. Take the mid-point $(x_1 + a_1)/2$. Because we want the value of a_2 to be strictly larger than x_1 as well, we pick a sufficiently small real number ϵ so that the interval

$$[x_1 + \epsilon, (x_1 + a_1)/2]$$

is nonempty. By the property that rational numbers are dense, we can find a rational number a_2 that is within the above interval. Recursively, after a_k is chosen, we pick a_{k+1} to be a rational number that is strictly between x_1 and $(x_1 + a_k)/2$. Throughout this process, the value of a_n is strictly larger than x_1 for all n, and a_n is a decreasing sequence converging to x_1 from the right.

Similar argument guarantee that we can choose the sequence b_n's with the required properties.

Then, we can write the open interval (x_1, x_2) as a countable union

$$(x_1, x_2) = \bigcup_{n=1}^{\infty} [a_n, b_n].$$

Because $[a_n, b_n]$ are $\mathscr{F}$-measurable, the union (x_1, x_2) is also $\mathscr{F}$-measurable.

We note that when we take the union, the two boundary point x_1 and x_2 are not included.

Question 4. (16 points)

(a) Prove that if $\mathscr{F}_1$ and $\mathscr{F}_2$ are σ-algebras on a common sample space Ω, then the intersection $\mathscr{F}_1 \cap \mathscr{F}_2$ is also a σ-algebra. (The notation $\mathscr{F}_1 \cap \mathscr{F}_2$ means the collection consisting of sets that are member of both $\mathscr{F}_1$ and $\mathscr{F}_2$.)

(b) Give an example of two σ-algebras $\mathscr{F}_1$ and $\mathscr{F}_2$ on a common sample space Ω such that the union $\mathscr{F}_1 \cup \mathscr{F}_2$ is not a σ-algebra.

Solution. (a) We assume that $\mathscr{F}_1$ and $\mathscr{F}_2$ satisfy the axioms of σ-algebra, and show that $\mathscr{F}_1 \cap \mathscr{F}_2$ also satisfies the axioms.

- Because $\mathscr{F}_1$ and $\mathscr{F}_2$ are both σ-algebra, Ω belongs to both $\mathscr{F}_1$ and $\mathscr{F}_2$. Therefore Ω is a set in $\mathscr{F}_1 \cap \mathscr{F}_2$.
- Suppose A is a set in $\mathscr{F}_1 \cap \mathscr{F}_2$. Because A is an element in $\mathscr{F}_1$, A^c is an element in $\mathscr{F}_1$. On the other hand, because A is in $\mathscr{F}_2$, A^2 is in $\mathscr{F}_2$. Therefore A^c is in $\mathscr{F}_1 \cap \mathscr{F}_2$.

- Suppose $(A_n)_{n=1}^{\infty}$ is a sequence of sets in $\mathscr{F}_1 \cap \mathscr{F}_2$. For $i = 1, 2$, we have $A_n \in \mathscr{F}_i$ for all n. Since $\mathscr{F}_i$ is a σ-algebra, we have $\cup_{n=1}^{\infty} A_n \in \mathscr{F}_i$. Because it holds for $i = 1$ and $i = 2$, we obtain

$$\bigcup_{n=1}^{\infty} A_n \in \mathscr{F}_1 \cap \mathscr{F}_2.$$

(b) We construct a finite example with $\Omega = \{a, b, c\}$. The first σ-algebra $\mathscr{F}_1$ is

$$\mathscr{F}_1 \triangleq \{\emptyset, \{a\}, \{b, c\}, \Omega\},$$

and the second σ-algebra $\mathscr{F}_2$ is

$$\mathscr{F}_2 \triangleq \{\emptyset, \{c\}, \{a, b\}, \Omega\}.$$

The union $\mathscr{F}_1 \cup \mathscr{F}_2$ is the following collection of sets,

$$\mathscr{F}_1 \cup \mathscr{F}_2 = \{\emptyset, \{a\}, \{c\}, \{a, b\}, \{b, c\}, \Omega\}.$$

This collection of sets fails to be a σ-algebra, because it is not closed under union. The union of $\{a\}$ and $\{c\}$ is not a member of $\mathscr{F}_1 \cup \mathscr{F}_2$.

(An alternate argument is to use the fact that a σ-algebra over a finite sample space Ω must contains 2^k subsets, for some integer k. The union $\mathscr{F}_1 \cup \mathscr{F}_2$ contains 6 subsets, and 6 is not a power of 2, hence it cannot be a σ-algebra.)

Question 5. (24 points) John plays a game an infinitely number of times. The game is designed in such a way that the outcomes at different times are independent, and the probability of winning the n-th game is $1/n$, where $n = 1, 2, 3, \ldots$. Hence, if John plays this game, he will win the first game with probability 1, the second game with probability 1/2, and the third game with probability 1/3, and so on.

(a) For $n \geq 1$, let E_n be the event that John win at both time n and time $n+1$. Are the sequence of events $(E_n)_{n=1}^{\infty}$ independent?

(b) Determine the probability that John wins an infinitely number of times.

(c) Find the probability that John wins in two consecutive games an infinite number of times.

Solution. (a) The events E_n's are not independent. Consider the events E_2 and E_3. The probabilities of events E_2 and E_3 are

$$P(E_2) = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$$
$$P(E_3) = \frac{1}{3} \cdot \frac{1}{4} = \frac{1}{12}$$

The union $E_2 \cap E_3$ is the event that John wins at time 2, 3 and 4. This happens with probability

$$P(E_2 \cap E_3) = \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{4} = \frac{1}{24}.$$

It is clear that

$$P(E_2 \cap E_3) \neq P(E_2) \cdot P(E_3).$$

Therefore E_2 and E_3 are dependent events. If any two events in the sequence are dependent, the whole sequence of events is dependent.

(b) We check that the infinite series

$$\sum_{n=1}^{\infty} P(\text{John wins at the } n\text{-th game}) = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

is divergent. By the second Borel-Cantelli lemma, John wins an infinitely number of times with probability 1.

(c) The events E_n's are dependent, but in the first Borel-Cantelli lemma, it is not required that the events are independent. The sum of probabilities

$$\sum_{n=1}^{\infty} P(E_n) = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} < \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

is finite. Therefore, John wins in two consecutive games infinitely often with probability 0.

(We show in part (b) that

$$P(\text{infinitely many wins}) = 1,$$

but in part (c) we see that

$$P(\text{infinitely many consecutive wins}) = 0.$$

This is possible. We can also check that, with probability 1, almost all winning game will follow by a losing game.)

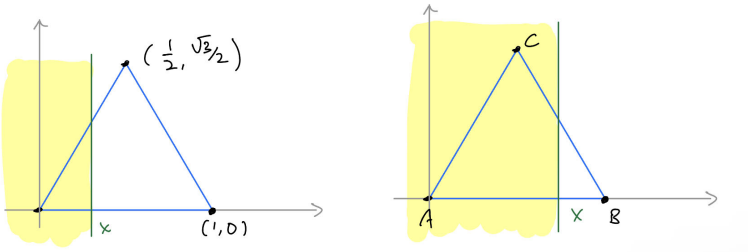
Question 6. (24 points) Let T represent the boundary of equilateral triangle with vertices $A = (0,0)$, $B = (1,0)$ and $C = (1/2, \sqrt{3}/2)$. (The interior of the triangle is not included in T.) We generate a random point (X,Y) uniformly on T. This means that, with probability $1/3$, the random point (X,Y) is uniformly chosen on the line segment AB, and with probability $1/3$, (X,Y) is uniformly chosen on BC, and with probability $1/3$, (X,Y) is uniformly chosen on CA.

(a) Describe the distribution of the x-coordinate X of the random point.

(b) Describe the distribution of the y-coordinate Y of the random point.

(c) Compute the expectation of X and the expectation of Y.

Solution. (a) From the geometry, we can see that the distribution of X should be uniform between 0 and 1.



Let's verify it formally by computing $P(X \le x)$. The probability $P(X \le x)$ is the length of the portion of the triangle in the shaded area, divided by the total length 3. When $0 \le x \le 1/2$, the region to the left of vertical line with x-coordinate x is shown in the above figure. It includes the portion of the horizontal sides from $(0,0)$ to $(x,0)$, and the slanted side from $(0,0)$ to $(x, \sqrt{3}x)$. The probability that a random point on T will fall on this region is

$$P(X \le x) = \frac{1}{3}\big(x + \frac{x}{\cos(\pi/3)}\big) = x.$$

When $1/2 \le x \le 1$. The region to the left of the vertical line with x-coordinate x include the whole line segment. In this case, we have

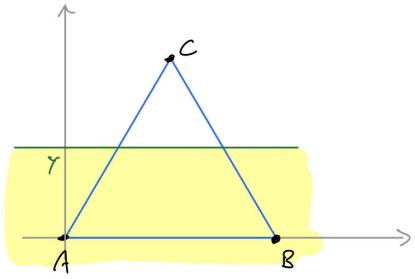
$$P(X \le x) = \frac{1}{3}\big(1 + x + \frac{x - 1/2}{\cos(\pi/3)}\big) = x.$$

Hence, for $0 \le x \le 1$, $P(X \le x) = x$. The random variable X is thus uniformly distributed between 0 and 1.

(b) The cdf of Y has a discontinuity at the point $y = 0$. When $y < 0$, we certainly has $P(Y \le y) = 0$. However, the region $Y \le y$ include the line segment AB. As a result, we have

$$P(Y \le 0) = 1/3$$

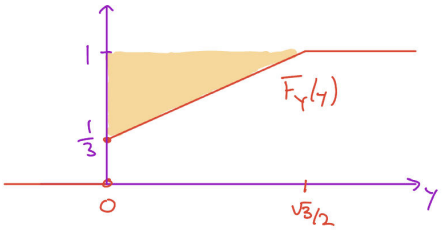
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When $0 < y \le \sqrt{3}/2$, the portion of the triangle that is in the above shaded area consists of three parts: the whole line segment AB, part of AC and part of BC. The probability that a random point chosen from the triangle will be in the shaded region is

$$P(Y \le y) = \frac{1}{3}\big(1 + 2\frac{y}{\sin(\pi/3))}\big) = \frac{1}{3}\big(1 + \frac{4y}{\sqrt{3}}\big)$$

The cumulative distribution function of Y is shown in the graph below



(c) Because X is uniformly distributed, the expectation of X can be computed easily and is equal to $1/2$.

For nonnegative random variable, we can use the formula

$$E[Y] = \int 1 - F_Y(y)\, dy$$

to compute the expectation of Y. This is the area of the shaded region in orange color in the above graph. The expectation of Y is

$$E[Y] = \frac{\frac{2}{3} \cdot \frac{\sqrt{3}}{2}}{2} = \frac{\sqrt{3}}{6}.$$