STA 4001 Stochastic Processes Spring 2020
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Final Exam
Sept 1

1. (20 points) Consider a discrete-time Markov chain $X = \{X_n : n = 0, 1, 2, \ldots\}$ with state space $S = \{0, 1, 2\}$ and transition matrix given by

$$P = \begin{bmatrix} 0 & 1/3 & 2/3 \\ 2/3 & 1/3 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

   (a) (5 points) Draw the transition diagram. For each state, determine whether it is transient, null recurrent, or positive recurrent.

   (b) (4 points) Suppose the initial distribution is given by $\mathbb{P}(X_0 = 0) = \mathbb{P}(X_0 = 1) = 1/2$. Find the distribution of X_2.

   (c) (3 points) Show that all states of this Markov chain communicate with each other.

   (d) (4 points) Suppose the process is now in state 2. Find the expected number of transitions until the process is in state 2 again.

   (e) (4 points) Find $\mathbb{P}(X_3 = 2, X_2 \neq 0, X_1 \neq 0 | X_0 = 1)$.

Solution.

   (a) This is a irreducible MC with finite state, each state is p.r.

   (b) $\mathbb{P}(X_2 = \cdot) = \frac{1}{2}(\mathbb{P}(X_2 = \cdot | X_0 = 0) + \mathbb{P}(X_2 = \cdot | X_0 = 1)) = \frac{1}{2}$(the first row of$P^2$+the second row of$P^2$) = $\frac{1}{2}([\frac{2}{9}, \frac{7}{9}, 0] + [\frac{2}{9}, \frac{3}{9}, \frac{4}{9}]) = [\frac{2}{9}, \frac{5}{9}, \frac{2}{9}]$.

   (c) As we can see from the transition diagram, any state is accessible from other states within finite steps.

   (d) Solving the balance equation yields $\pi = [\frac{6}{19}, \frac{9}{19}, \frac{4}{19}]$. Since it is a p.r. irreducible MC, $\mathbb{E}[T_2] = \frac{1}{\pi_2} = \frac{19}{4}$

   (e) Notice that $P_{12} = P_{22} = 0$, therefore $\mathbb{P}(X_3 = 2, X_2 \neq 0) = \mathbb{P}(X_3 = 2, X_2 = 1, 2) = 0$. $\mathbb{P}(X_3 = 2, X_2 \neq 0, X_1 \neq 0 | X_0 = 1) = \mathbb{P}(X_3 = 2, X_2 \neq 0 | X_1 \neq 0)\mathbb{P}(X_1 \neq 0 | X_0 = 1) = 0$.

2. (20 points) Let $X = \{X(t), t \geq 0\}$ be a continuous time Markov chain on $S = \{1, 2, 3\}$.The generator matrx is

$$G = \begin{bmatrix} -2 & 2 & 0 \\ 1 & -3 & 2 \\ 0 & 4 & -4 \end{bmatrix}.$$

   (a) (5 points) If we start from state 1, compute $P'_{1,i}(0+)$ for all $i \in S$, where $P'_{1,i}(t) = \dfrac{d}{dt}\mathbb{P}[X(t) = i | X(0) = 1]$.

   (b) (5 points) Find a stationary distribution of the CTMC X.

   (c) (5 points) Compute the jump matrix J and holding time rates $\{\lambda(i), i \in S\}$. Is the CTMC positive recurrent?

   (d) (5 points) Specify the inputs for a uniformization Markov chain by specifying a jump matrix $\hat{J}$ and holding time rates $\{\hat{\lambda}(i), i \in S\}$.

Solution.

   (a) Since $P'(0+) = P(0)G = G$, $P'_{1,\cdot}(0+) = [-2, 2, 0]$

(b) By solving the equation $\pi G = 0$ we can get $\pi = [\frac{1}{4}, \frac{1}{2}, \frac{1}{4}]$

(c) By $\lambda(i) = -G_{ii}$ and $J_{ij} = \frac{G_{ij}}{\lambda(i)}$, we have $[\lambda(1), \lambda(2), \lambda(3)] = [2, 3, 4]$ and

$$J = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{3} & 0 & \frac{2}{3} \\ 0 & 1 & 0 \end{bmatrix}.$$

The CTMC is p.r.

(d) First we choose $\hat{\lambda} = \max_i \lambda(i) = 4$, and then $\hat{J} = I + \frac{1}{\hat{\lambda}} G =$

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix}.$$

The CTMC X can be represented via $X(t) = Y_{N(t)}$, where $\{Y_n, n \geq 0\}$ is a DTMC with transition matrix $\hat{J}$ and $\{N(t), t \geq 0\}$ is a Poisson process with rate $\hat{\lambda}$.

3. (24 points) Jobs arrive into a system consisting of two servers, according to a Poisson process with rate $\lambda = 10$ (jobs per second). Service times are exponentially distributed. A job arriving is sent to server 1 with probability $p = 40\%$, which can serve jobs with rate $\mu_1 = 5$ (jobs per second). Otherwise, it is sent to server 2, which can serve jobs with rate $\mu_2 = 10$ (jobs per second). Each server has its own queue, and can fit up to $K = 3$ total jobs (i.e. 1 in service and $K - 1$ waiting). Jobs arriving to a full server get rejected. Suppose the sytem is empty intially.

(a) (4 points) Let $(X_1(t), X_2(t))$ be the queue size (including jobs in service) of two servers, respectively, at time t. Are $X_1(t)$ and $X_2(t)$ independent?

(b) (6 points) Find the long-run-average number of jobs in the system.

(c) (6 points) What is the throughput of this system (i.e. jobs served per second)?

(d) (4 points) Find the long-run-average time in the system of each customer that enters the system.

(e) (4 points) If a job sees 2 jobs at server 1 and 3 jobs at server 2 upon arrival, what is the expected amount of time the job spends in the system (waiting time + service time)?

Solution.

(a) Yes. $X_1(t)$ and $X_2(t)$ are independent. This is two separate M/M/1/K queues, one with input rate $p\lambda$, the other with $(1-p)\lambda$.

(b) For $X_1(\cdot)$, apply "cut method" to derive a recursive formula for the stationary probabilities as:

$$\pi_1 = \frac{p\lambda}{\mu_1}\pi_0$$

$$\pi_n = \left(\frac{p\lambda}{\mu_1}\right)^n \pi_0, \quad n = 1, 2, \ldots, K$$

Since $\sum_{n=0}^{K} \left(\frac{p\lambda}{\mu_1}\right)^n \pi_0 = 1$, we have

$$\pi_0 = \frac{1 - (p\lambda/\mu_1)}{1 - (p\lambda/\mu_1)^{K+1}} = \frac{125}{369}$$

Denote $\rho_1 = \frac{p\lambda}{\mu_1} = \frac{4}{5}$,

$$L_1 = \sum_{n=0}^{K} n\rho_1^n \pi_0 = \pi_0\rho_1 \sum_{n=0}^{K} n\rho_1^{n-1} = \pi_0\rho_1 \frac{d}{d\rho_1}\left(\sum_{n=0}^{K} \rho_1^n\right) = \pi_0\rho_1 \frac{1 - (K+1)\rho_1^K + K\rho_1^{K+1}}{(1-\rho_1)^2}$$

In the same manner,denote $\rho_2 = \dfrac{(1-p)\lambda}{\mu_2} = \dfrac{3}{5}$, $\hat{\pi}_0 = \dfrac{1-\rho_2}{1-\rho_2^{K+1}} = \dfrac{125}{272}$,

$$L_2 = \hat{\pi}_0 \rho_2 \frac{1-(K+1)\rho_2^K + K\rho_2^{K+1}}{(1-\rho_2)^2}$$

Then, the long run average number of jobs in the system is $L = L_1 + L_2 = 1103/518 \approx 2.129$.

(c) From server 1 and server 2, new jobs are rejected at a rate of $p\lambda\pi_K$ and $(1-p)\lambda\hat{\pi}_K$ respectively. So the system throughput is

$$\hat{\lambda} = \lambda\left(p(1-\pi_K) + (1-p)(1-\hat{\pi}_K)\right) \approx 8.71$$

(d) By Little's law, we get

$$W = L/\hat{\lambda} \approx 0.24.$$

(e) By the memoryless of service times, the expected amount of time the job spends in the system will be

$$p(2 \times 1/\mu_1 + 1/\mu_1) + (1-p) \times 0 = 0.24.$$

4. (20 points) Jack plays the betting game as follows. At each time n, he roll a fair dice (with values $1, 2 \ldots, 6$), if the result of the dice is less than 6, he loses \$1 (in this case he set $X_n = -1$), otherwise he win \$$x$ (in this case he set $X_n = +1$). Let M_n be the total equities of Jack and assume he starts the game without money ($M_0 = 0$).

   (a) (3 points) Find a formula for M_n in terms of $X_1, \ldots, X_n$ up to time n.

   (b) (7 points) Find the value of x for which M_n is a martingale with respect to $X_1, \ldots, X_n$.

   (c) (10 points) Let now x be the value found in part (b). Suppose that Jack stops playing either when $M_n \geq 100$ or when $M_n \leq -10$. Compute the expected equities of Jack when he stops playing.

   Solution.

   (a) $M_n = \sum_{i=1}^n (-1(X_i = -1) + x1(X_i = +1)) = \dfrac{x+1}{2}\sum_{i=1}^n X_i + \dfrac{x-1}{2}n$, $\forall n \geq 1$.

   (b) Since $\mathbb{E}(|M_n|) < \infty$, if M_n is a martingale, we have

$$\mathbb{E}(M_{n+1}|X_1, \ldots, X_n) = \frac{x+1}{2}\sum_{i=1}^n X_i + \frac{x-1}{2}(n+1) + \frac{x+1}{2}\mathbb{E}(X_{n+1}) = M_n$$

   That is, $\dfrac{x+1}{2}\mathbb{E}(X_{n+1}) + \dfrac{x-1}{2} = -\dfrac{x+1}{3} + \dfrac{x-1}{2} = 0$, $x = 5$.

   (c) Let $\tau = \inf\{n \geq 1; M_n \geq 100 \text{ or } M_n \leq -10\}$, this is a stopping time. Since $|M_n| \leq 100$, when $n \leq \tau$ and $\tau < \infty$ w.p. 1, we can apply optional sampling theorem,

$$\mathbb{E}(M_\tau) = \mathbb{E}(M_0) = 0.$$

5. (16 points) Consider a production system consisting of three single-server stations in series. A job needs to be processed at station 1 first, and then at station 2. After being processed at station 2, a job has $p_1 = 10\%$ probability going back to station 1 for rework and $1 - p_1$ probability continuing onto station 3. After being processed at station 3, the job leaves the production system as a finished product. Assume that the processing times of jobs at each station are i.i.d, having exponential distribution, regardless of the history of the jobs. The average processing times at stations 1, 2 and 3 are $m_1 = 0.8$, $m_2 = 0.9$ and $m_3 = 1$ hours, respectively. The Management decides to adopt the make-to-stock policy, using the CONWIP job release policy. Recall that when the system is operating CONWIP policy only a job leaving the system triggers a new job to be released to station 1. Let N be the CONWIP level.

(a) (10 points) For $N = 2$, compute the throughput of the production system. What is the average time in system per job?

(b) (6 points) Is it possible to double the throughput? If so, what N is needed to achieve it? What is the corresponding average time in system per job?

Solution.
(a) Set λ_i to be the throughput at station i. The traffic equations are then

$$\begin{aligned}\lambda_1 &= 0.1\lambda_2 + \lambda_3\\ \lambda_2 &= \lambda_1\\ \lambda_3 &= 0.9\lambda_2\end{aligned}$$

Setting $\lambda_2 = 1$ and solving this system gives $\lambda_1 = 1$ and $\lambda_3 = 0.9$. Note that these are not the actual throughput values - we will address this later. Using $\rho_i = \lambda_i \times m_i$ give us that $\rho_1 = 0.8$, $\rho_2 = .9$ and $\rho_3 = 0.9$. Now define$X_i(t)$ to be the number of items at station i at time t. Given that the WIP level is 2 we can use this to determine the true throughput and utilization rates,

$$\begin{aligned}P(X_1(\infty) = 2, X_2(\infty) = 0, X_3(\infty) = 0) &= C\rho_1^2\rho_2^0\rho_3^0\\ P(X_1(\infty) = 1, X_2(\infty) = 1, X_3(\infty) = 0) &= C\rho_1^1\rho_2^1\rho_3^0\\ P(X_1(\infty) = 1, X_2(\infty) = 0, X_3(\infty) = 1) &= C\rho_1^1\rho_2^0\rho_3^1\\ P(X_1(\infty) = 0, X_2(\infty) = 1, X_3(\infty) = 1) &= C\rho_1^0\rho_2^1\rho_3^1\\ P(X_1(\infty) = 0, X_2(\infty) = 2, X_3(\infty) = 0) &= C\rho_1^0\rho_2^2\rho_3^0\\ P(X_1(\infty) = 0, X_2(\infty) = 0, X_3(\infty) = 2) &= C\rho_1^0\rho_2^0\rho_3^2\end{aligned}$$

These must sum to 1 so we can solve for C to get C = 1/4.51. Having solved for C we can determine the fraction of time that machine 1 is idle,

$$P(X_1(\infty) = 0, X_2(\infty) = 1, X_3(\infty) = 1) = 0.81C$$

$$P(X_1(\infty) = 0, X_2(\infty) = 2, X_3(\infty) = 0) = 0.81C$$

$$P(X_1(\infty) = 0, X_2(\infty) = 0, X_3(\infty) = 2) = 0.81C$$

Summing these gives the fraction of the time that machine 1 is idle, which is 243/451. Thus the (true) utilization of machine 1 is $\rho_1^* = 1 - 243/451 = 208/451$. Backsolving for ρ_3^* (the true throughput) gives us that $\rho_3^* = 234/451$.
Consequently the throughput is $\lambda_3^* = 234/451$. Finally we use Little's law to solve for W = L/(λ_3^*) (with L = 2) to get $W = 451/117$ hours per job.

(b) To double the throughput we would need to double the throughput of machine 2, which requires doubling the utilization of machine 2. However,$\rho_2^* = 9/8\rho_1^* > 1/2$ so that we can't double the utilization to possible to double the throughput.