Final Examination

1. (10 points) Let $X_1, \ldots, X_n$ be n independent random variables with a same Poisson distribution with rate parameter $\lambda > 0$. Define $S_n = X_1 + \ldots + X_n$.

   (a) (3 points) Find $\mathbb{E}(S_n|X_1)$.

   (b) (3 points) Find $\mathbb{E}(X_1|S_n)$.

   (c) (4 points) Let s be a non-negative integer and consider the conditional distribution of $(X_1, \ldots, X_n)$ given $S_n = s$.

   (c1) What is the domain of definition of this conditional distribution?

   (c2) Determine the p.m.f. of the conditional distribution.

2. (20 points)

   Let $Z_0, Z_1, Z_2, \ldots$ be independent, identically distributed random variables with common p.m.f.:

k	0	1	2	3
p_k	0.1	0.3	0.2	0.4

   For $n \geq 0$, define $X_n = \min(Z_0, Z_1, \ldots, Z_n)$.

   (a) (4 points) Give the transition matrix P of the Markov chain (X_n).

   (b) (2 points) Calculate $\mathbb{P}(X_3 = 2, X_2 = 1, X_1 = 2|X_0 = 3)$

   (c) (2 points) Calculate $\mathbb{P}(X_3 = 0, X_2 = 1, X_1 = 2|X_0 = 3)$.

   (d) (2 points) Find the communication classes of the chain.

   (e) (2 points) Find the recurrent and transient states of the chain.

   (f) (2 points) What are the periods of the states?

   (g) (3 points) Does the chain have a stationary (invariant) distribution?

   (h) (3 points) Find the limiting probabilities of the chain.

3. (10 points) Let X_1, X_2 and X_3 be independent exponential random variables with respective rates λ_j, $j = 1, \ldots, 3$.

   (a) (5 points) Show that

$$\mathbb{P}(X_1 < X_2 < X_3) = \frac{\lambda_1 \lambda_2}{(\lambda_2 + \lambda_3)(\lambda_1 + \lambda_2 + \lambda_3)}.$$

   (b) (5 points) What is the value of this probability when all the λ_j's are equal? Can you find this value by a different method?

4. (15 points) Let $\{N_t\}_{t \geq 0}$ be a Poisson process with rate 3.

   (a) (2 points) Calculate $P(N_4 = 3)$.

   (b) (2 points) Calculate $P(N_4 = 3, N_6 = 6, N_{10} = 12)$.

   (c) (2 points) Calculate $P(N_6 = 6|N_4 = 3)$.

   (d) (2 points) Calculate $P(N_6 = 6|N_{10} = 12)$.

   (e) (2 points) Determine $E(N_{10}|N_6)$ and $E(N_{10}|N_4, N_6)$.

   (f) (5 points) Show that conditional on $N_{10} = n$, N_6 has the binomial distribution $\mathcal{B}(n, 3/5)$. Determine $E(N_6|N_{10})$.

5. (10 points) An insurance company has two insurance portfolios, namely A and B. Claims in portfolio A occur in accordance with a Poisson process with mean 3 per year. Claims in portfolio B occur in accordance with a Poisson process with mean 5 per year. The two processes are independent.

   (a) (5 points) Calculate the probability that the 3rd claim in portfolio A occurs before the 3rd claim in portfolio B in a certain year.

   (b) (5 points) Given that exactly eight claims had occurred in the first year, find the expected arrival time of the first claim.

6. (15 points) Consider a special two-station open Jackson network. Each station has a single server and an infinite capacity waiting area. External arrivals to station 1 follow a Poisson process with rate α, and service times at station i are i.i.d, following exponential distribution with rate μ_i, $i = 1, 2$. After being served at station 1, each customer goes to station 2, and after being served at station 2, the customer goes back to station 1 with probability 20% and leaves the system with probability 80%.

   Assume that

   $$(1.25)\alpha < \mu_1 \text{ and } (1.25)\alpha < \mu_2.$$

   Let $X_i(t)$ be the number of customers at station i at time t, $i = 1, 2$. Then $X = \{(X_1(t), X_2(t)), t \geq 0\}$ is a continuous time Markov chain.

   (a) (5 points) Prove that X is irreducible.

   (b) (10 points) Prove that X has a stationary distribution by the following three steps:

      (i) write the flow balance equation for *each state* according to the four cases: $(0, 0)$, $(n_1, 0)$, $(0, n_2)$ and (n_1, n_2) where $n_1 > 0$ and $n_2 > 0$;

      (ii) guess a form of the stationary distribution, and

      (iii) verify that the guessed stationary distribution does satisfy the flow balance equation for each state.

7. (20 points) Let $B(t), t \geq 0$ be a standard Brownian motion.

   (a) (5 points) Determine $P(B(2.1) > 1.7 | B(1.6) = 0.5)$

   (b) (5 points) Determine $P(B(1.6) \leq 0.5 | B(2.1) = 1.7)$

   (c) (5 points) Show that for any $0 < s < t$, $B(t)$ and $B(s) - \frac{s}{t}B(t)$ are independent.

   (d) (5 points) Let $M(t) = \max_{0 \leq s \leq t} B(s)$. Determine $\mathbb{E}[M(1)]$.

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