MAT2042 Midterm Review

Liu Zhaocheng

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Chapter 1

Review of Chapter 1-4

1.1 Vector Spaces

Definition 1.1. A vector space V over a field F ($\mathbb{R}$ or $\mathbb{C}$) consists of a set on which two operations, addition ($+ : V \times V \to V$) and scalar multiplication ($\cdot : F \times V \to V$), are defined, such that the following conditions hold for all $x, y, z \in V$ and $a, b \in F$:

1. (VS1) $x + y = y + x$
2. (VS2) $(x + y) + z = x + (y + z)$
3. (VS3) There exists an element $0 \in V$ such that $0 + x = x$
4. (VS4) For each $x \in V$, there exists $y \in V$ such that $x + y = 0$
5. (VS5) $1 \cdot x = x$ for the multiplicative identity $1 \in F$
6. (VS6) $(ab)x = a(bx)$
7. (VS7) $a(x + y) = ax + ay$
8. (VS8) $(a + b)x = ax + bx$

Example 1.2. $\mathbb{R}^2, \mathbb{R}^n, \mathcal{F}(\mathbb{R}), \mathcal{C}(\mathbb{R}), \mathcal{P}_n(\mathbb{R})$ are vector spaces. $\mathcal{P}_n(\mathbb{R})$ is the set of all polynomials with real coefficients of degree $\leq n$. $\mathcal{C}(\mathbb{R})$ is the set of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$.

1.2 Subspaces

Definition 1.3. A subset $W \subseteq V$ is a subspace if W is a vector space with the same operations (addition and scalar multiplication) as defined on V.

Remark 1.4. To verify whether a set W is a subspace, we must check:

1. The zero vector $0 \in W$.
2. W is closed under addition.
3. W is closed under scalar multiplication.

Proposition 1.5.
- *Any intersection of subspaces is a subspace.*
- *The union of subspaces is not necessarily a subspace.*

1.3 Linear Combinations and Span

Definition 1.6. Let $v, u_1, \ldots, u_k \in V$. If there exist scalars $a_1, \ldots, a_k \in F$ such that $v = a_1u_1 + \cdots + a_ku_k$, then v is called a linear combination of $u_1, \ldots, u_k$.

Definition 1.7. Let $S \subseteq V$. The span of S, denoted span(S), is the set consisting of all linear combinations of the vectors in S.

Proposition 1.8.
- *If $S = \emptyset$, then span$(S) = \{0\}$.*
- *span(S) is the smallest subspace of V that contains S.*

1.4 Linear Independence

Definition 1.9. A set of vectors $\{u_1, \ldots, u_n\}$ is linearly dependent if there exist scalars $a_1, \ldots, a_n \in F$, not all zero, such that $\sum_{i=1}^{n} a_iu_i = 0$. Otherwise, the set is linearly independent.

Proposition 1.10.
- *Let $S_1 \subseteq S_2 \subseteq V$. If S_1 is linearly dependent, then S_2 is linearly dependent. If S_2 is linearly independent, then S_1 is linearly independent.*
- *Suppose $S \subseteq V$ is linearly independent. Then $S \cup \{v\}$ is linearly dependent if and only if $v \in$ span(S).*

1.5 Bases

Definition 1.11. A basis for a vector space V is a linearly independent subset that also spans V.

Theorem 1.12.
- *$\beta = \{u_1, \ldots, u_n\}$ is a basis for V if and only if every $v \in V$ can be uniquely expressed as a linear combination of vectors in β.*
- *If $V =$ span(S) for a finite set S, then some subset of S is a basis for V.*

Theorem 1.13 (Replacement Theorem)**.** *Let G be a set of n vectors that spans V, and let L be a linearly independent subset of m vectors. Then $m \leq n$ and there exists a subset $H \subseteq G$ containing $n - m$ vectors such that span$(L \cup H) = V$.*

1.6 Dimension

Definition 1.14. A vector space V is finite-dimensional if it has a basis consisting of a finite number of vectors. The number of vectors in the basis is the dimension of V, denoted $\dim(V)$.

Corollary 1.15. *If $\dim(V) = n$, then:*

- *Any generating set must contain at least n vectors, and a generating set of n vectors is a basis.*
- *Any linearly independent set contains at most n vectors, and a linearly independent set of n vectors is a basis.*
- *Any linearly independent set can be extended to a basis.*

Proposition 1.16. *Let $W \subseteq V$ be a subspace of a finite-dimensional space V. Then* $\dim(W) \leq \dim(V)$. *If* $\dim(W) = \dim(V)$, *then $W = V$.*

Definition 1.17. Let W_1, W_2 be subspaces of V. The sum is $W_1 + W_2 = \{w_1 + w_2 : w_1 \in W_1, w_2 \in W_2\}$.

Proposition 1.18. *Let V be a finite-dimensional vector space, and W_1, W_2 be subspaces. Then:*

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$$

1.7 Linear Transformations, Null Spaces, and Ranges

Definition 1.19. Let V and W be vector spaces over the same field F. A function $T : V \to W$ is a linear transformation if for all $x, y \in V$ and $c \in F$, we have:

1. $T(x + y) = T(x) + T(y)$
2. $T(cx) = cT(x)$

Definition 1.20. Let V and W be vector spaces, and $T : V \to W$ be linear.

- The null space (or kernel) of T, denoted $N(T)$, is the set $\{x \in V : T(x) = 0\}$.
- The range (or image) of T, denoted $R(T)$, is the set $\{T(x) : x \in V\}$.

Proposition 1.21. *$N(T)$ is a subspace of V, and $R(T)$ is a subspace of W.*

Theorem 1.22 (Dimension Theorem / Rank-Nullity Theorem)**.** *Let V and W be vector spaces, and $T : V \to W$ be linear. If V is finite-dimensional, then:*

$$\dim(N(T)) + \dim(R(T)) = \dim(V)$$

The dimension of $N(T)$ is called the nullity, and the dimension of $R(T)$ is called the rank.

Theorem 1.23. *Let $T : V \to W$ be a linear transformation.*

- *T is injective (one-to-one) if and only if $N(T) = \{0\}$.*
- *T is surjective (onto) if and only if $R(T) = W$.*

Corollary 1.24. *1. T is one-to-one(injective) $\iff N(T) = \{0\}$.*

2. If $\dim V = \dim W$, then the following statements are equivalent:

(a) T is one-to-one

(b) T is onto(surjective)

(c) rank$(T) = \dim V$

Proposition 1.25. *Let $\beta = \{v_1, \ldots, v_n\}$ be a basis for V, and let $\{w_1, \ldots, w_n\}$ be an arbitrary subset of n vectors in W. Then, there exists a unique linear transformation $T : V \to W$ such that $T(v_j) = w_j$ for $j = 1, \ldots, n$.*

1.8 Matrix Representation

Definition 1.26. Let V be a finite-dimensional vector space. An ordered basis for V is a basis $(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ where the order of vectors is fixed.

Definition 1.27. Let $\beta = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$ be an ordered basis for V. For any $x \in V$, the coordinate vector of x relative to β, denoted $[x]_\beta$, is the column vector of scalars $(c_1, \ldots, c_n)^T$ such that $x = \sum_{i=1}^n c_i \mathbf{v}_i$.

Theorem 1.28 (Matrix representation of a linear transformation)**.** *Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there exists a unique $m \times n$ matrix A called the matrix representation of T such that $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$.*

Furthermore, the j-th column of A is $T(\mathbf{e}_j)$, where $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is the standard basis for $\mathbb{R}^n$.

Definition 1.29. The $m \times n$ matrix $A = (a_{ij})_{m \times n}$ is the matrix representation of the transformation T with respect to the ordered bases β and γ, we write:

$$A = [T]_\beta^\gamma.$$

(When $V = W$ and $\beta = \gamma$, $[T]_\beta^\gamma = [T]_\beta$.)

Corollary 1.30. *Furthermore,*

$$[T(x)]_\gamma = [T]_\beta^\gamma [x]_\beta \quad \forall\, x \in V.$$

Moreover, the matrix $[T]_\beta^\gamma$ is constructed by taking $[T(v_j)]_\gamma$ as its j-th column:

$$[T]_\beta^\gamma = \begin{pmatrix} [T(v_1)]_\gamma & [T(v_2)]_\gamma & \ldots & [T(v_n)]_\gamma \end{pmatrix}.$$

Definition 1.31. Let A be an $m \times n$ matrix with entries from a field F. We denote by $\mathbf{L}_A$ the mapping $\mathbf{L}_A : F^n \to F^m$ defined by $\mathbf{L}_A(x) = Ax$ for each column vector $x \in F^n$. We call $\mathbf{L}_A$ a left-multiplication transformation.

Definition 1.32. Let V and W be vector spaces over a field $\mathbb{F}$. We define the set of linear transformations from V to W as:

$$\mathcal{L}(V, W) = \{T : V \to W \mid T \text{ is a linear transformation}\}.$$

Proposition 1.33. *$\mathcal{L}(V, W)$ is a vector space.*

Theorem 1.34. *If V and W are finite-dimensional with $\dim V = n$ and $\dim W = m$, then:*

$$\dim \mathcal{L}(V, W) = m \cdot n.$$

Furthermore, the following diagram commutes:

$$\begin{array}{ccc} V & \xrightarrow{\quad T \quad} & W \\ {\scriptstyle \phi_\beta}\downarrow & & \downarrow{\scriptstyle \phi_\gamma} \\ \mathbb{F}^n & \xrightarrow[L_A]{} & \mathbb{F}^m \end{array}$$

where $\Phi_\beta(v) = [v]_\beta$ and $\Phi_\gamma(w) = [w]_\gamma$ are called coordinate maps.

Corollary 1.35.

- $L_A \circ \Phi_\beta = \Phi_\gamma \circ T$.
- $\mathcal{L}(V, W) \cong W_{m \times n}(\mathbb{R})$.

1.9 Composition and Matrix Multiplication

Definition 1.36. Let A be an $m \times n$ matrix and B be an $n \times p$ matrix. We define the **product** of A and B, denoted AB, to be the $m \times p$ matrix such that

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

for $1 \leq i \leq m,\ 1 \leq j \leq p$.

Theorem 1.37. *Let $T : V \to W$ and $U : W \to Z$ be linear transformations. Then the composition $UT : V \to Z$ is a linear transformation. Furthermore, if the spaces are finite-dimensional with ordered bases α, β, γ, then:*

$$[UT]_\alpha^\gamma = [U]_\beta^\gamma [T]_\alpha^\beta$$

Definition 1.38. The transpose of A, denoted A^t, is the $n \times m$ matrix in $M_{n,m}(\mathbb{F})$ where:

$$(A^t)_{i,j} = A_{j,i}$$

for all integers $1 \leq i \leq n$ and $1 \leq j \leq m$.

Proposition 1.39. *For any matrices $A \in M_{i,j}(\mathbb{F})$ and $B \in M_{j,k}(\mathbb{F})$, then:*

$$(AB)^t = B^t A^t$$

1.10 Invertibility and Isomorphisms

Definition 1.40. A linear transformation $T : V \to W$ is invertible if there exists a map $U : W \to V$ such that $TU = I_W$ and $UT = I_V$. U is called the inverse of T and denoted T^{-1}.

Definition 1.41. Let A be an $n \times n$ matrix. A is **invertible** if there exists an $n \times n$ matrix B such that $AB = BA = I$ (where I denotes the $n \times n$ identity matrix).

Proposition 1.42. *1. T is invertible if and only if T is injective (one-to-one) and surjective (onto).*

2. If T is invertible, then its inverse T^{-1} is also a linear transformation.

3. If $T_1 : V \to W$ and $T_2 : W \to Z$ are both invertible linear transformations, then the composition $T_2 \circ T_1 : V \to Z$ is also invertible, and $(T_2 \circ T_1)^{-1} = T_1^{-1} \circ T_2^{-1}$.

4. If T is invertible, then T^{-1} is also invertible, and $(T^{-1})^{-1} = T$.

5. If V or W is finite-dimensional and T is invertible, then $\dim V = \dim W$.

Theorem 1.43. *A linear transformation $T : V \to W$ between finite-dimensional spaces is invertible if and only if its matrix representation $[T]_\beta^\gamma$ is an invertible matrix. In this case, $[T^{-1}]_\gamma^\beta = ([T]_\beta^\gamma)^{-1}$.*

Definition 1.44. A vector space V is isomorphic to W (denoted $V \cong W$) if there exists an invertible linear transformation $T : V \to W$. Such a T is called an isomorphism.

Theorem 1.45. *Two finite-dimensional vector spaces V and W are isomorphic if and only if* $\dim(V) = \dim(W)$.

1.11 Change of Basis

Definition 1.46. Let β and γ be two ordered bases for V. The change of coordinate matrix from γ to β is the matrix $Q = [I]_\gamma^\beta$. It satisfies $[x]_\beta = Q[x]_\gamma$.

Theorem 1.47. *(1) Let V be a finite-dimensional vector space with two ordered bases β and γ. Then*

(i) $Q = [I]_\gamma^\beta$ is invertible.

(ii) $[v]_\beta = [I]_\gamma^\beta [v]_\gamma$

(2) Let $T : V \to V$ be a linear transformation. Then,

$$[T]_\gamma = Q^{-1}[T]_\beta Q = \left([I]_\gamma^\beta\right)^{-1} [T]_\beta [I]_\gamma^\beta$$

Equivalently, we write $[T]_\beta = Q[T]_\gamma Q^{-1} = [I]_\gamma^\beta [T] ([I]_\gamma^\beta)^{-1}$

1.12 Elementary Matrix Operations and Elementary Matrices

Definition 1.48. Let A be an $m \times n$ matrix. The following are the three types of elementary row operations:

1. Interchanging two rows.
2. Multiplying a row by a non-zero scalar.
3. Adding a multiple of one row to another row.

Elementary column operations are defined analogously.

Definition 1.49. An $n \times n$ elementary matrix is a matrix obtained by performing a single elementary row operation on the $n \times n$ identity matrix I_n.

Proposition 1.50. *Elementary matrices are invertible, and their inverses are also elementary matrices of the same type.*

Theorem 1.51. *Let A be an $m \times n$ matrix. If an elementary row operation is performed on A to yield a matrix B, then $B = EA$, where E is the corresponding $m \times m$ elementary matrix.*

1.13 The Rank of a Matrix and Matrix Inverses

Definition 1.52. The rank of a matrix A, denoted $\text{rank}(A)$, is the dimension of the range of the linear transformation $L_A : F^n \to F^m$ defined by $L_A(x) = Ax$.

Proposition 1.53. *1. Let $A \in \text{M}_{m\times n}$, $P \in \text{M}_{m\times m}$, $Q \in \text{M}_{n\times n}$. P and Q are two invertible matrices. Then:* $\text{rank}(PA) = \text{rank}(A), \quad \text{rank}(AQ) = \text{rank}(A)$

2. $\text{rank}(A)$ *is the maximum number of linearly independent column vectors of A.*

Theorem 1.54. *Let $A \in M_{m\times n}(\mathbb{F})$ of rank r. Then $r \leq m$ and $r \leq n$. Moreover, by a finite number of elementary matrix operations (row/column), A can be transformed into:*

$$D = \begin{pmatrix} I_r & \mathbf{0}_{r\times(n-r)} \\ \mathbf{0}_{(m-r)\times r} & \mathbf{0}_{(m-r)\times(n-r)} \end{pmatrix}$$

Corollary 1.55. *1. Let $A \in M_{m\times n}(\mathbb{F})$ be of rank r. Then $\exists$ an invertible matrix $B \in M_{m\times m}(\mathbb{F})$ and an invertible matrix $C \in M_{n\times n}(\mathbb{F})$ such that $BAC = D$, where*

$$\underbrace{E_p \cdots E_1}_{B} A \underbrace{E'_1 \cdots E'_q}_{C} = D \quad \text{and} \quad D = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}.$$

2. rank(A) = rank(A^t)

3. rank(A) = is the maximum number of linearly independent row vectors.

4. Every invertible matrix is a product of elementary matrices.

1.14 Systems of Linear Equations

Definition 1.56. A system $Ax = b$ is called homogeneous if $b = 0$, and non-homogeneous if $b \neq 0$.

Theorem 1.57. *The set of solutions to a homogeneous system $Ax = 0$ is a subspace of F^n with dimension $n - rank(A)$.*

Proposition 1.58. *Let K be the set of all solutions to the nonhomogeneous system $Ax = b$. Then the solution set can be expressed as:*

$$K = K_h + \{s\}$$

where $s \in K$ is any particular solution to $Ax = b$.

Theorem 1.59. *Let K be the set of solutions to $Ax = b$, and K_H be the set of solutions to the corresponding homogeneous system $Ax = 0$. If s is any particular solution to $Ax = b$, then $K = \{s + k : k \in K_H\}$.*

Definition 1.60. A matrix is in Reduced Row Echelon Form (RREF) if:

1. Any row consisting entirely of zeros lies at the bottom.
2. The first non-zero entry in each non-zero row is 1 (called a pivot).
3. Each pivot is the only non-zero entry in its column.
4. Each pivot lies to the right of the pivot in the row above it.

Theorem 1.61. *Every matrix can be transformed into a unique RREF matrix by elementary row operations.*

1.15 Determinants

Definition 1.62. For a 1×1 matrix $A = [a_{11}]$, the determinant is $\det(A) = a_{11}$. For an $n \times n$ matrix A, the determinant is defined inductively as:

$$\det(A) = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det(A_{1j})$$

where A_{1j} is the $(n-1) \times (n-1)$ matrix obtained by deleting the first row and the j-th column of A.

Theorem 1.63. *The determinant of an $n \times n$ matrix can be computed by cofactor expansion along any row or any column.*

Proposition 1.64. *The determinant of an upper triangular or lower triangular matrix is the product of its diagonal entries. Specifically,* $\det(I) = 1$.

Theorem 1.65. *The effect of elementary row operations on the determinant is as follows:*

1. *Interchanging two rows changes the sign of the determinant.*
2. *Multiplying a row by a non-zero scalar k multiplies the determinant by k.*
3. *Adding a multiple of one row to another row does not change the determinant.*

Theorem 1.66. *An $n \times n$ matrix A is invertible if and only if* $\det(A) \neq 0$.

Proposition 1.67 (Properties of Determinant)**.** *Let $A \in M_{n\times n}$.*

1. $\det(A)$ *is a linear function of each row when the remaining rows are held fixed.*
2. *If A has a row consisting entirely of zeros* $\implies \det(A) = 0$.
3. $\det(A)$ *can be evaluated by cofactor expansion along any row.*
4. $\det(A) = 0$ *if A has two identical rows.*
5. *If B is obtained from A by interchanging any two rows of A* $\implies \det(B) = -\det(A)$.
6. *If B is obtained from A by adding a multiple of one row of A to another row* $\implies \det(B) = \det(A)$.
7. *If rank$(A) < n$* $\implies \det(A) = 0$.
8. $\det(AB) = \det(A) \cdot \det(B)$.
9. $\det(A^{-1}) = \frac{1}{\det(A)}$.
10. $\det(A^t) = \det(A)$.

Corollary 1.68. *Similar matrices have the same determinant.*

Chapter 2

Small Problems

1. How to prove or disprove whether a set is a vector space?
2. How to prove or disprove whether a subset is a subspace of a given vector space?
3. How to prove that a vector space is the direct sum of two subspaces?
4. How to clearly distinguish the conceptual difference between the sum $W_1 + W_2$ and the intersection $W_1 \cap W_2$ of subspaces?
5. How to use Gaussian elimination to determine the existence of a linear combination?
6. How to prove that two vector spaces or sets are identical?
7. How to verify whether a set of vectors satisfies the definition of linear independence?
8. How to determine whether a set constitutes a basis for the space?
9. What is the Replacement Theorem, and what is its purpose?
10. How to calculate the dimension of the sum of two subspaces?
11. How to prove that a mapping between two vector spaces is a linear transformation?
12. How to determine the null space and the range of a linear transformation?
13. What is the Dimension Theorem?
14. How to prove whether a linear transformation is injective (1-1) or surjective (onto)?
15. How to construct the matrix representation of a linear transformation relative to given ordered bases?
16. What is the relationship between the composition of maps and matrix multiplication?
17. How to determine if a linear transformation is an isomorphism between two spaces?
18. How to prove that two finite-dimensional vector spaces are isomorphic?
19. How to compute the change of coordinate matrix (transition matrix) between two bases?
20. How to show that matrix representations of the same operator under different bases are similar?

21. How to perform elementary row operations and identify their corresponding elementary matrices?
22. How to prove that elementary operations preserve the rank of a matrix?
23. How to determine the rank of a matrix using Gaussian elimination to find its RREF?
24. How to find the inverse of an invertible matrix using the augmented matrix $[A \mid I]$?
25. How to characterize the solution space of a homogeneous system of linear equations?
26. How to relate the consistency of $Ax = b$ to the rank of the coefficient matrix and the augmented matrix?
27. How to express the general solution of a non-homogeneous system as a sum of a particular solution and the homogeneous solution?
28. How to prove that an $n \times n$ matrix is invertible if and only if its rank is n?
29. How to prove that every invertible matrix can be expressed as a product of elementary matrices?
30. How to explain the geometric meaning of row rank being equal to column rank?
31. How to compute the determinant of an $n \times n$ matrix using cofactor expansion along any row or column?
32. How to describe the effect of each type of elementary row operation on the determinant?
33. How to use elementary row operations to transform a matrix into a triangular form to compute its determinant efficiently?
34. How to prove that a matrix is invertible if and only if its determinant is non-zero?
35. How to demonstrate that the determinant is a multi-linear function with respect to the rows of a matrix?
36. How to relate the determinant of a product of matrices to the product of their individual determinants?
37. How to prove that a matrix and its transpose always have the same determinant?
38. How to show that if a matrix has two identical rows, its determinant must be zero?
39. How to prove that similar matrices must have the same determinant?
40. How to interpret the determinant as a signed volume scaling factor of a linear transformation?

Chapter 3

Examples

Example 3.1. Let A and B be $n \times n$ matrices. Does $AB = BA$ always hold? If not, provide a counter-example and a brief geometric justification.

Example 3.2. Let $A, B \in M_{n\times n}(F)$ be invertible matrices such that $ABA^{-1} = B^{-1}$.

1. Prove that $A^2B = BA^2$.
2. Prove that $(AB)^2 = A^2$.

Example 3.3. Let $u, v,$ and w be distinct vectors in V. Prove that $\{u, v, w\}$ is linearly independent if and only if $\{u + v, u + w, v + w\}$ is linearly independent.

Example 3.4. Prove that if W_1 and W_2 are finite-dimensional subspaces of a vector space V, then the subspace $W_1 + W_2$ is finite-dimensional, and $\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$.

Example 3.5. Consider the plane $x-2y+z = 0$ in $\mathbb{R}^3$. Find the reflection of the point $A = (3, 0, 0)^T$ across this plane.

Example 3.6. Define $T : P_2(\mathbb{R}) \to M_{2\times2}(\mathbb{R})$ by $T(f) = \begin{pmatrix} f'(0) & 2f(1) \\ 0 & f''(3) \end{pmatrix}$. Let $\beta = \{1, x, x^2\}$ and $\alpha = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ be the standard ordered bases.

1. Compute the matrix representation $[T]^\alpha_\beta$.
2. Compute the coordinate vector $[T(f)]_\alpha$ for $f(x) = 3x^2 - 6x + 4$.
3. Suppose $T(p) = \begin{pmatrix} 2 & 10 \\ 0 & 4 \end{pmatrix}$. Use $[T]^\alpha_\beta$ to find the polynomial $p(x) \in P_2(\mathbb{R})$.

Example 3.7. Let V be the subset of $\mathbb{R}^5$ defined by the homogeneous linear equation:

$$V = \Big\{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 \mid x_1 + 7x_2 + 5x_3 - 4x_4 + 2x_5 = 0\Big\}$$

Consider the subset $S = \{s_1, s_2, s_3\} \subset \mathbb{R}^5$, where:

$$s_1 = (2, 0, 0, 1, 1), \quad s_2 = (1, 1, -2, -1, -1), \quad s_3 = (-5, 1, 0, 1, 1).$$

1. Verify that V is a vector space.
2. Show that $S \subseteq V$ and that S is linearly independent.
3. Extend S to a basis for V.

Example 3.8. Solve the following system of linear equations:

$$\begin{cases} x_1 - 2x_2 - x_3 = 1 \\ 2x_1 - 3x_2 + x_3 = 6 \\ 3x_1 - 5x_2 = 7 \\ x_1 + 5x_3 = 9 \end{cases}$$

Example 3.9. Evaluate the following determinant:

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{vmatrix}$$