MAT2060: Honours Mathematical Analysis
Assignment #1

Deadline: Sep. 21

1. Let A and B be nonempty bounded above subsets of $\mathbb{R}$, and let $A+B$ be the set of all sums $a+b$ where $a \in A$ and $b \in B$. Show that $\sup(A+B) = \sup A + \sup B$.

2. Find the sup, inf, max and min for the following sets

$$(a) \quad A = \left\{ \frac{1}{n} \,|\, n \in \mathbb{N} \right\}; \qquad (b) \quad B = \left\{ 1 - \frac{1}{n} \,|\, n \in \mathbb{N} \right\}.$$

3. Let X and Y be two sets and $f : X \to Y$ be a function. Show that for $A, B \subset X$,

(i) $f(A \cup B) = f(A) \cup f(B)$;

(ii) $f(A \cap B) \subset f(A) \cap f(B)$;

(iii) $f(A \cap B) = f(A) \cap f(B)$ provided that f is injective.

4. A sketch proof of the Schröder–Bernstein Theorem is given in the lecture notes. Complete the proof with details.

5. Show the following two sets have the same cardinality using the Schröder–Bernstein Theorem.

$$(a) \quad [0,1] \sim (0,1) \qquad (b) \quad [0,1] \sim [0,1] \times [0,1].$$

Hint. For (b), to find a 1–1 function from $[0,1] \times [0,1]$ to $[0,1]$, consider the decimal representation of real numbers.

6. Answer each of the following by establishing a 1–1 correspondence with a set of known cardinality.

(i) Is the set of all functions from {0,1} to $\mathbb{N}$ countable or uncountable?

(ii) Is the set of all functions from $\mathbb{N}$ to {0, 1} countable or uncountable?

7. Set $x_1 = 2$ and

$$x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n}, \qquad \forall n \in \mathbb{N}.$$

(i) Show that (x_n) is bounded below and decreasing.

(ii) Show that the (x_n) converges and find its limit.

(iii) Show that the set $A = \{x_n \mid n \in \mathbb{N}\}$ is a subset of $\mathbb{Q}$ and bounded below, but there is no greatest lower bound (inf) of A exists in $\mathbb{Q}$.

Remark:
(i) This example shows that $\mathbb{Q}$ is not complete, since the Least Upper Bound Property doesnot hold in $\mathbb{Q}$.
(ii) Recall that the Newton method to find a root of $f(x) = 0$ is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Thus, the sequence (x_n) in this example is given by the Newton's method for $f(x) = x^2 - 2$.

8. (i) Show that the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. **Hint.** Apply the Monotone Convergence Theorem to the sequences of partial sums.

(ii) Why the harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges? **Hint.** Show the sequences of partial sums is unbounded.

9. Assume the Bolzano–Weierstrass Theorem is true and use it to construct a proof of the Monotone Convergence Theorem.

10 (A simpler proof of the Heiner–Borel theorem)**.** Let $M > 0$ and $I = [-M, M]$ be a closed bounded interval.

(i) Show that any open cover of I has a finite subcover. Proof by contradiction and use the Nested Interval Property.

(ii) Assume $K \subset \mathbb{R}$ is closed and bounded, show that any open cover of K has a finite subcover.

11. Using the Bolzano–Weierstrass Theorem to prove the Cauchy Criterion: Any Cauchy sequence converges in $\mathbb{R}$.

12. Assume K is a compact set. Show that both $\sup K$ and $\inf K$ exist and they are elements of K, that is the $\max K$ and $\min K$ exits.

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