MAT2060: Honours Mathematical Analysis
Assignment 10

Deadline: Dec. 17

1. Find the area of the domain enclosed by the *lemniscate*

$$(x^2+y^2)^2 = a^2(x^2-y^2), \qquad a>0.$$

Hint. 1) Area of the region enclosed by the simple closed curve C is given by

$$\frac{1}{2}\int_C xdy - ydx.$$

2) Use the symmetry to reduce the problem to the first quadrant, and find a parametrization of C using the polar coordinates.

2. Consider the linear differential form

$$L = \frac{ydx - xdy}{x^2+y^2}$$

(i) Verify that L satisfies the integrability condition for $D = \mathbb{R}^2 \setminus \{(0,0)\}$. Why this does not necessarily imply that L is exact in D?

(ii) Let C be a simple smooth curve from $(1,0)$ to $(0,2)$ that lies entirely in the first quadrant (except for the initial and final points). Evaluate the line integral $\displaystyle\int_C L$.

3. (i) Evaluate the integral

$$\int_C \frac{e^y}{x^2+y^2}[(x\sin x + y\cos x)dx + (y\sin x - x\cos x)dy],$$

where C is the unit circle $x^2+y^2=1$ with positive orientation.

(ii) Show that

$$\int_0^{2\pi} e^{r\cos\theta}\cos(r\sin\theta)d\theta = 2\pi.$$

4. Let C be a smooth Jordan curve with positive orientation. Evaluate the Gauss integral

$$\int_C \frac{\cos(\boldsymbol{r},\boldsymbol{n})}{r}ds,$$

where $\boldsymbol{n}$ is the unit outward normal vector at point $(\xi,\eta)\in C$, $\boldsymbol{r}$ is the vector $(\xi-x,\eta-y)$ with $(x,y)\notin C$, r is the length $\|\boldsymbol{r}\|$, and $(\boldsymbol{r},\boldsymbol{n})$ denotes the angle between $\boldsymbol{r}$ and $\boldsymbol{n}$.

5. Suppose D is a bounded closed region in $\mathbb{R}^2$ with boundary being a smooth Jordan curve. Assume u is C^2 in the interior of D and C^1 in D, u is nonconstant and $u = 0$ on the boundary ∂D. Show that

$$I = \iint_D u \cdot \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) < 0.$$

Hint. Consider the integral

$$\int_{\partial D} -u \frac{\partial u}{\partial y} dx + u \frac{\partial u}{\partial x} dy.$$

6. For a continuous function $f(x)$, show that

$$\iiint\limits_{x^2+y^2+z^2 \le 1} f(ax + by + cz) dx dy dz = \pi \int_{-1}^{1} (1 - u^2) f(\sqrt{a^2 + b^2 + c^2}\, u) du$$

7 (Surface of revolution)**.** Let $C : (x(t), y(t))$, $a \le t \le b$ and consider the surface by rotating C about the y-axis. Show the area of the surface of revolution is given by

$$2\pi \int_a^b |x(t)| \sqrt{x'(t)^2 + y'(t)^2} dt\,.$$

8. Evaluate

$$\iint\limits_{x+y+z=t} f(x, y, z) dS,$$

where

$$f(x, y, z) = \begin{cases} 1 - x^2 - y^2 - z^2 & \text{when } x^2 + y^2 + z^2 \le 1 \\ 0 & \text{when } x^2 + y^2 + z^2 > 1. \end{cases}$$

9 (Green's identities)**.** Let $\Omega \subset \mathbb{R}^3$ be a bounded closed region with boundary $\partial\Omega$ being a smooth surface. Assume $u \in C^2(\Omega)$ and $v \in C^1(\Omega)$.

(i) Show that

$$\iiint_\Omega (v \Delta u + \nabla v \cdot \nabla u) dV = \iint_{\partial\Omega} v (\nabla u \cdot \boldsymbol{n}) dS,$$

where $\Delta u = u_{xx} + u_{yy} + u_{zz}$, ∇u is the gradient of u, and $\boldsymbol{n}$ is the outward normal direction of the boundary $\partial\Omega$.

(ii) If both u and v are C^2 in Ω, show that

$$\iiint_\Omega (v \Delta u - u \Delta v) dV = \iint_{\partial\Omega} \left(v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \right) dS,$$

where $\frac{\partial u}{\partial n} = \nabla u \cdot \boldsymbol{n}$ is the directional derivative of u along $\boldsymbol{n}$.

(iii) If u is a harmonic function, that is it is a solution of the Laplace equation $\Delta u = 0$. Show that

$$\iint_{\partial\Omega} \frac{\partial u}{\partial n} dS = 0.$$

10. Assume Σ is a closed smooth surface, $\boldsymbol{v}$ is an arbitrary direction. Show that

$$\iint_{\Sigma} \boldsymbol{n} \cdot \boldsymbol{v}\, dS = 0,$$

where $\boldsymbol{n}$ is the outward normal direction.

11. Assume $\Omega \subset \mathbb{R}^3$ is a bounded domain and its surface $\partial\Omega$ is smooth, and the point $(x_0, y_0, z_0) \notin \Omega$. Let

$$\boldsymbol{r} = \frac{(x - x_0, y - y_0, z - z_0)}{r}, \qquad r = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}.$$

Show that

$$2 \iiint_{\Omega} \frac{dxdydz}{r} = \iint_{\partial\Omega} \boldsymbol{r} \cdot \boldsymbol{n}\, dS\,,$$

where $\boldsymbol{n}$ is outward surface normal.

12. Let Σ denote a simple closed smooth surface and let $\boldsymbol{n}$ denote its outward surface normal.

(i) Assume $\boldsymbol{F}$ is a twice continuously differentiable vector field. Show that

$$\mathrm{div}(\mathrm{curl}\boldsymbol{F}) = 0$$

and that

$$(*) \qquad \iint_{\Sigma} \mathrm{curl}\boldsymbol{F} \cdot \boldsymbol{n}\, dS = 0.$$

(ii) Assume $\boldsymbol{F}$ is only continuously differentiable. Show that $(*)$ still holds.
Hint. Make use of Stokes' theorem.

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