MAT2060: Honours Mathematical Analysis
Assignment #2

Deadline: Sep 30

1. Show that the sequence

$$\sqrt{2}, \quad \sqrt{2+\sqrt{2}}, \quad \sqrt{2+\sqrt{2+\sqrt{2}}}, \quad \cdots,$$

is convergent and find its limit.

2. For a bounded sequence $\{x_n\}$, the Bolzano–Weierstrass Theorem says that there exists a convergent subsequence. Let E be the set of real numbers s such that $x_{n_k} \to s$ for some subsequence $\{x_{n_k}\}$. Show that

$$\limsup_{n\to\infty} x_n = \sup E \qquad \text{and} \qquad \liminf_{n\to\infty} x_n = \inf E.$$

3. For the following sequences, find their upper and lower limits.

$$\text{(i)} \quad \{(-1)^n\}_{n=1}^{\infty}, \qquad \text{(ii)} \quad \{(-1)^n n\}_{n=1}^{\infty}, \qquad \text{(iii)} \quad \left\{(-1)^n \frac{1}{n}\right\}_{n=1}^{\infty}.$$

4. Show that a sequence $\{x_n\}$ is convergent if and only if $\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n$. In this case, all three share the same value.

5. Let (x_n) be a bounded sequence. Show that if all convergent subsequence of (x_n) converge to the same limit, then x_n converges to the same limit.

6 (Order Properties for Upper and Lower Limits)**.** Assume there exists $M \in \mathbb{N}$ such that $x_n \le y_n$ for each $n \ge M$. Show that

$$\liminf_{n\to\infty} x_n \le \liminf_{n\to\infty} y_n, \qquad \limsup_{n\to\infty} x_n \le \limsup_{n\to\infty} y_n.$$

7. Assume $0 \le x_{n+m} \le x_n + x_m$ for all $n, m \in \mathbb{N}$. Show that the sequence $\left\{\frac{x_n}{n}\right\}$ converges. **Hint.** Apply the result about upper and lower limits in the above two problems.

8. Assume $\lim_{n\to\infty} x_n = A$. Show that

$$\lim_{n\to\infty} \frac{\frac{1}{2}x_1 + \frac{2}{3}x_2 + \cdots + \frac{n}{n+1}x_n}{n} = A.$$

9. Assume $x_n > 0$ for every $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} \dfrac{x_{n+1}}{x_n} = \ell < \infty$. Show that $\lim\limits_{n\to\infty} \sqrt[n]{x_n} = \ell$.

10. Assume $x_n > 0$ for every $n \in \mathbb{N}$. Show that

$$\limsup_{n\to\infty} \sqrt[n]{x_n} \le \limsup_{n\to\infty} \frac{x_{n+1}}{x_n}.$$

11. Prove the intermediate value theorem by the Least Upper Bound Principle.

12. Assume $f : \mathbb{R} \to \mathbb{R}$ is an increasing function. Show that all the discontinuities of f are jumps and there are at most countable many of jumps.

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