MAT2060: Honours Mathematical Analysis
Assignment #3

Deadline: Oct. 10

1 (LUBP implies HB)**.** Consider the special case where K is a closed interval. Let $\{O_\lambda \,|\, \lambda \in \Lambda\}$ be an open cover for $[a, b]$ and define S to be the set of all $x \in [a, b]$ such that $[a, x]$ has a finite subcover from $\{O_\lambda \,|\, \lambda \in \Lambda\}$.

(i) Argue that S is nonempty and bounded, and thus $s = \sup S$ exists.

(ii) Now show $s = b$, which implies $[a, b]$ has a finite subcover.

(iii) Finally, prove the theorem for an arbitrary closed and bounded set K.

2. (i) Prove that $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

(ii) Does this result about closures extend to infinite unions of sets?

3. A dual notion to the closure of a set is the *interior* of a set. The interior of E is denoted E° and is defined as

$$E^\circ = \{x \in E \,|\, \text{there exists } V_\epsilon(x) \subset E\}.$$

Results about closures and interiors possess a useful symmetry.

(i) Show that E is open if and only if $E^\circ = E$.
Remark. We have shown that E is closed if and only if $\overline{E} = E$.

(ii) Show that $(\overline{E})^c = (E^c)^\circ$ and $(E^\circ)^c = \overline{E^c}$.

4 (HB implies BW)**.**

(i) Using Heine–Borel Theorem to prove that every bounded infinite set in $\mathbb{R}^n$ has a limit point. Therefore, every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence (BW).

(ii) Let (S, d) be a metric space and $K \subset S$ be a subset. Show that if K is compact, then K is also sequentially compact.

Remark. In the lecture, we have shown that if K is sequentially compact, then K is compact. Therefore, in a metric space, the sequentially compactness and compactness are equivalent. However, this is not necessarily true in a topological space.

5 (Upper and Lower Limits)**.** As in the case of sequential limits, we have the upper and lower limits for a function,

$$\limsup_{x\to c} f(x) := \lim_{\delta\to 0^+} \sup_{0<|x-c|<\delta} f(x),$$

$$\liminf_{x\to c} f(x) := \lim_{\delta\to 0^+} \inf_{0<|x-c|<\delta} f(x).$$

Show that $\lim_{x\to c} f(x)$ exists if and only if both $\limsup_{x\to c} f(x)$ and $\liminf_{x\to c} f(x)$ exist and they are equal to each other.

6. Assume $h : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and let $K = \{x \,|\, h(x) = 0\}$. Show that K is a closed set.
Remark. The set K is called a level set. The result can be extended to continuous functions among two metric spaces.

7. Let $F \subset \mathbb{R}$ be a nonempty closed set and define $g(x) = \inf\{|x - a| : a \in F\}$. Show that g is continuous on all of $\mathbb{R}$ and $g(x) \neq 0$ for all $x \notin F$.

8. Let (S_1, d_1) and (S_2, d_2) are two metric spaces, and let $f : S_1 \to S_2$ be a function.

Recall the theorem "A function that is continuous on a compact set K is uniformly continuous on K." Provide a proof by the definition "$K \subset \mathbb{R}$ is compact if every open cover of K has a finite subcover."

9. (i) Assume that g is defined on an open interval (a, c) and it is known to be uniformly continuous on $(a, b]$ and $[b, c)$, where $a < b < c$. Prove that g is uniformly continuous on (a, c).

(ii) Prove that $f(x) = \sqrt{x}$ is uniformly continuous on $[0, \infty)$.

(iii) Show that $f(x) = x^p$ with $p \in \mathbb{R}$ is uniformly continuous on $(0, \infty)$ if and only if $0 \leq p \leq 1$.

(iv) Assume $f(x)$ is a continuous function defined on $[0, \infty)$, and assume that $\lim_{x\to\infty} f(x) = L \in \mathbb{R}$. Show that $f(x)$ is uniformly continuous on $[0, \infty)$.

10 (Continuous Extension Theorem)**.**

(i) Show that a uniformly continuous function preserves Cauchy sequences; that is, if $f : A \to \mathbb{R}$ is uniformly continuous and $\{x_n\} \subset A$ is a Cauchy sequence, then show $f(x_n)$ is a Cauchy sequence.

(ii) Let g be a continuous function on the open interval (a, b). Prove that g is uniformly continuous on (a, b) if and only if it is possible to define values $g(a)$ and $g(b)$ at the endpoints so that the extended function g is continuous on $[a, b]$. (In the forward direction, first produce candidates for $g(a)$ and $g(b)$, and then show the extended g is continuous.)

11. Let S be a topological space which is connected. Show that the only sets that are both open and closed are $\emptyset$ and S.

12. In the lecture, we gave the sketch of the proof that a continuous function preserves connectedness. Complete this proof:

Let $f : S_1 \to S_2$ be a continuous functions among two metric spaces (or, even more general, two topological spaces) and let S_1 be connected. We want to show that $f(S_1)$ is also connected.

Let $f(S_1) = A \cup B$ with $A \neq \emptyset$, $B \neq \emptyset$ and $A \cap B = \emptyset$. Denote $C := f^{-1}(A)$ and $D := f^{-1}(B)$.

(i) Show that $f^{-1}(A \cup B) = C \cup D$, $C \neq \emptyset$, $D \neq \emptyset$, $C \cap D = \emptyset$.

(ii) Show that $\overline{f(C)} \supset f(\overline{C})$, and similarly $\overline{f(D)} \supset f(\overline{D})$.

(iii) Show that $f(S_1)$ is connected.

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