MAT2060: Honours Mathematical Analysis
Assignment #4

Deadline: Oct. 18

1. (i) Show that any open set on $\mathbb{R}$ can be written as a disjoint union of at most countable open intervals.

(ii) Show that any open set on $\mathbb{R}^n$ can be written as a union of at most countable closed cells (R_k) with (R_k°) being mutually disjoint. Here a closed cell R in $\mathbb{R}^n$ is defined as

$$R = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n].$$

(iii) Show that any connected set in $\mathbb{R}$ must be an interval.

2. A topological space S is called *path-connected* if any two points in S can be jointed by a curve γ joining these two points; that is, for any $p, q \in S$, there exists a continuous function $\gamma : [0, 1] \to S$ such that $\gamma(0) = p$ and $\gamma(1) = q$.

(i) Show that any path-connected space is connected.

(ii) Show that the topologist's sine curve is not path-connected.

Remark. We have shown that the topologist's sine curve is connected. Thus path-connectedness is a stronger notion than connectedness. Also recall that a *domain* $\Omega \in \mathbb{R}^n$ is a path-connected open set.

3 (Continuous function preserves compactness)**.** Let S_1 and S_2 be two topological space, and let $f : S_1 \to S_2$ be a continuous function.

(i) Recall that in a topological space, a sequence (x_n) converges to x if for any open subset U with $x \in U$, there exists $N \in \mathbb{N}$ such that $x_n \in U$ for $n \geq N$. Show that continuous functions preserve convergence sequences, that is if $x_n \to x$ in S_1, then $f(x_n) \to f(x)$ in S_2.

(ii) Show that if $K \subset S_1$ is sequentially compact, then $f(K)$ is sequentially compact.

(iii) Show that if $K \subset S_1$ is compact then $f(K)$ is compact.

Remark. In the lectures, we show that if S_1 and S_2 are metric spaces and f is continuous on S_1, then $f(K)$ is compact provided K is compact. There, we argue with sequentially compactness. Recall that, in metric spaces, sequential compactness and compactness are equivalent. However, in general topological spaces, sequential compactness and compactness are not equivalent.

This problem shows that continuous functions preserve compactness as well as sequentially compactness.

4. Recall the definition of uniform continuous functions on $A \subset \mathbb{R}$:

> For any $\varepsilon > 0$, there exists $\delta > 0$ such that
>
> $$|f(x) - f(y)| < \varepsilon \qquad \forall |x - y| < \delta \ \ (\text{of course, } \ x, y \in A).$$

Show that the negation is: f is not uniformly continuous on A if

> there exist $\varepsilon > 0$ and two sequences (x_k) and (y_k) in A, such that
>
> $$|x_k - y_k| \to 0 \qquad \text{but} \qquad |f(x_k) - f(y_k)| \geq \varepsilon_0 \quad \forall k \in \mathbb{N}.$$

Using this criterion to show that the following functions are not uniform continuous on $(0, 1)$.

$$(a) \quad f(x) = \sin \frac{1}{x}; \qquad (b) \quad g(x) = \ln x; \qquad (c) \quad h(x) = \frac{1}{1 - x}.$$

5. Let $f : [0, 1] \to \mathbb{R}$ be continuous with $f(0) = f(1)$.

(i) Show that there must exist $x, y \in [0, 1]$ satisfying $|x - y| = 1/2$ and $f(x) = f(y)$.

(ii) Show that for each $n \in \mathbb{N}$ there exist $x_n, y_n \in [0, 1]$ with $|x_n - y_n| = 1/n$ and $f(x_n) = f(y_n)$.

(iii) If $h \in (0, 1/2)$ is not of the form $1/n$, there does not necessarily exist $|x - y| = h$ satisfying $f(x) = f(y)$. Provide an example that illustrates this using $h = 2/5$.

6 (Inverse functions)**.** If a function $f : A \to \mathbb{R}$ is one-to-one, then we can define the inverse function f^{-1} on the range of f in the natural way: $f^{-1}(y) = x$ where $y = f(x)$. Show that if f is continuous on an bounded interval $[a, b]$ and one-to-one, then f^{-1} is also continuous.

7 (Baire's theorem)**.** Let S be a complete metric space. Show that

(i) If (F_n) is a sequence of closed subsets of S with $F_n^\circ = \emptyset$ for each n. Then $(\bigcup_{n=1}^\infty F_n)^\circ = \emptyset$.

This can be paraphrased as "*A countable union of nowhere dense closed sets is still nowhere dense.*"

(ii) If (G_n) is a sequence of open subsets of S with $\overline{G_n} = S$ for each n. Then $\overline{\bigcap_{n=1}^\infty G_n} = S$.

This can be paraphrased as "*A countable intersection of dense open sets is still dense.*"

Hint. First prove that (i) and (ii) are equivalent by noting that $(\overline{E})^c = (E^c)^\circ$ and $(E^\circ)^c = \overline{E^c}$, which is Problem 3(ii) in Assignment 3.

Then prove (ii) as follows. Let $O \subset S$ be an open subset, w.t.s. $O \cap (\bigcap_{n=1}^{\infty} G_n) \neq \emptyset$.

There exists $x_1 \in O \cap G_1$ (why?), thus there exists $r_1 > 0$ such that $\overline{B(x_1, r_1)} \subset O \cap G_1$ (why?)

Now, there exist x_2 and r_2 with $0 < r_2 < r_1/2$ such that $\overline{B(x_2, r_2)} \subset B(x_1, r_1) \cap G_2$ (why?)

Iteratively, there exist x_n and r_n with $0 < r_n < r_{n-1}/2$ such that $\overline{B(x_n, r_n)} \subset B(x_{n-1}, r_{n-1}) \cap G_n$.

Then, (x_n) is a Cauchy sequence (why?) and thus converges (why? – here is the only place requires S to be complete.) Let $x_n \to x$, then $x \in \overline{B(x_n, r_n)} \subset G_n$ for each n (why?), and thus $x \in O$ (why?)

(iii) Show that (i) implies that the complete metric space S is of the second category.

Remark. This is Baire's category theorem we proved in lecture: "*A complete metric space is of the second category.* Part (i) and (ii) are also referred as Baire's category theorem.

8 (Corollary of Baire's category theorem)**.** Let S be a metric space and let $S = \bigcup_{n=1}^{\infty} F_n$, where each F_n is closed.

Show that

(i) If $F_n^{\circ} = \emptyset$ for each n, then S is not complete.

(ii) If S is complete, then there exists $n_0 \in \mathbb{N}$ such that $(F_{n_0})^{\circ} \neq \emptyset$.

9. Given a countable set $A = \{a_1, a_2, a_3, \dots\}$, define $f(a_n) = 1/n$ and $f(x) = 0$ for all $x \notin A$. Find D_f.

10. Let S be a topological space. Recall that a set $A \subset S$ is called an F_σ set if we can write $A = \bigcup_{n=1}^{\infty} F_n$ where each F_n is closed, and a set $B \subset S$ is called a G_δ set if we can write $B = \bigcap_{n=1}^{\infty} G_n$ where each G_n is open.

(i) Show that A is F_σ if and only if A^c is G_δ.

(ii) Show that in $\mathbb{R}$, the set $\mathbb{Q}^c$ is not F_σ and thus $\mathbb{Q}$ is not G_δ.
Hint. Use Problem 8(ii).

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