MAT2060: Honours Mathematical Analysis
Assignment #5

Deadline: Oct. 27

1. Let K be a compact subset of $\mathbb{R}^n$ and let $f \in C(K)$. Show that there exists a sequence of functions (f_j) of $C^\infty(\mathbb{R}^n)$ such that $f_j(x) \to f(x)$ uniformly on K as $j \to \infty$.

2. In Problem 1, assume $f \in C^k(K)$ for some $k \in \mathbb{Z}_+ := \{0\} \cup \mathbb{N}$. Show that the sequence (f_j) in Problem 1 convergence in $C^k(K)$, that is,

$$\sum_{p=1}^{k} \sum_{|\alpha|=p} \sup_{x \in K} |\partial^\alpha f_j(x) - \partial^\alpha f(x)| \to 0 \qquad \text{as } j \to \infty,$$

where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$ is a multiple index, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, and $\partial^\alpha f = \frac{\partial^{|\alpha|} f}{\partial^{\alpha_1} x_1 \cdots \partial^{\alpha_n} x_n}$ is a partial derivative of f of $|\alpha|$-order.

3. Let V_n denote the volume of the unit ball $B(0,1)$ in $\mathbb{R}^n$; thus the volume of the ball $B(0,R) = V_n R^n$. Consider the function

$$\tilde{\rho}_\delta(x) = \begin{cases} \frac{1}{V_n \delta^n} & \text{if } x \in B(0,\delta), \\ 0, & \text{otherwise.} \end{cases}$$

and define

$$\tilde{f}_\delta(x) = (\tilde{\rho}_\delta * f) = \int_{\mathbb{R}^n} \tilde{\rho}_\delta(x-y) f(y) dy.$$

Show that

(i) $\tilde{f}_\delta(x) \to f(x)$ uniformly on any compact subset of $\mathbb{R}^n$ as $\delta \to 0^+$.

(ii) Suppose $f \in C^k(\mathbb{R}^n)$ for some $k \in \mathbb{Z}_+ := \{0\} \cup \mathbb{N}$, then $\tilde{f}_\delta \in C^{k+1}(\mathbb{R}^n)$.

Remark. Compare this problem with the mollifier.

4 (Partition of unity)**.**

(i) Let $K \subset U \subset \mathbb{R}^n$ with K being compact and U being open. Show that there exists a function $h \in C^\infty(\mathbb{R}^n)$ such that

$$h(x) = 1 \quad \forall x \in K, \qquad \text{and} \qquad 0 \le h(x) \le 1 \quad \forall x \in \mathbb{R}^n,$$

and the support of h, $\operatorname{supp} h$, is a compact subset of U.

(ii) Let $K \subset \mathbb{R}^n$ be compact and let $U_1, U_2, \ldots, U_N$ be a finite open cover of K. (Recall that any open cover of the compact K has a finite subcover). Show that there exists another open cover $V_1, V_2, \ldots, V_N$ such that every V_j is bounded and $\overline{V_j} \subset U_j$.

(iii) Let $K \subset \mathbb{R}^n$ be compact and let $U_1, U_2, \ldots, U_N$ be a finite open cover of K. Show that there exist functions $h_1(x), h_2(x), \ldots, h_N(x)$ such that

(1) $h_j \in C^\infty(\mathbb{R}^n)$ and $\operatorname{supp} h_j \in U_j$ $(j = 1, 2, \ldots, N)$.

(2) $0 \le h_j(x) \le 1$ for all $x \in \mathbb{R}^n$ and for $j = 1, 2, \ldots, N$;

(3) $\displaystyle\sum_{j=1}^{N} h_j(x) = 1$ for all $x \in K$.

Hint. Use part (ii) and part (i) to construct $e_j(x)$, $1 \le j \le N$. Let $e \in C^\infty(\mathbb{R}^n)$ be such that

$$e(x) = 1 \quad \forall x \in K, \qquad 0 \le e(x) \le 1 \qquad \operatorname{supp} e \subset \bigcup_{j=1}^{N} V_j$$

and define

$$h_j(x) = \begin{cases} \dfrac{e(x)e_j(x)}{e_1(x) + \cdots + e_N(x)}, & \text{if } x \in \displaystyle\bigcup_{j=1}^{N} V_j, \\ 0 & \text{otherwise.} \end{cases}$$

5 (Convergence in $\mathcal{C}(S)$)**.** Let S be a compact metric space and $\mathcal{C}(S)$ denote the space of all continuous functions on S. Let (f_n) be sequence of $\mathcal{C}(S)$ and $f \in \mathcal{C}(S)$. In the lecture, we have shown that

$$f_n \to f \quad \text{in } \mathcal{C}(S) \qquad \Longleftrightarrow \qquad f_n(x) \to f(x) \quad \text{uniformly on } S.$$

We have also shown that $\mathcal{C}(S)$ is complete, that is any Cauchy sequence converges.

In what follows, let $S = [0, 1]$ and consider $f_n(x) = x^n$.

(i) Find the limit $f(x)$ of $f_n(x)$ as $n \to \infty$. Is this convergence uniform on $[0, 1]$? Why?

(ii) Show that (f_n) is NOT a Cauchy sequence in $\mathcal{C}(S)$ by examine the definition of Cauchy sequences.

6. Observe that if a and b are real numbers, then

$$\max\{a, b\} = \frac{(a + b) + |a - b|}{2} \quad \text{and} \quad \min\{a, b\} = \frac{(a + b) - |a - b|}{2}.$$

(i) Show that if $f_1, f_2, \ldots, f_n$ are continuous functions, then

$$g(x) = \max\{f_1(x), f_2(x), \ldots, f_n(x)\}$$

and

$$h(x) = \min\{f_1(x), f_2(x), \ldots, f_n(x)\}$$

are continuous functions.

Hint. A simple approach for h is by noting that

$$h(x) = -\max\{-f_1(x), -f_2(x), \dots, -f_n(x)\}.$$

(ii) Let's explore whether the result in (i) extends to the infinite case. For each $n \in \mathbb{N}$, define f_n on $\mathbb{R}$ by

$$f_n(x) = \begin{cases} 1 & \text{if } |x| > 1/n \\ n|x| & \text{if } |x| \le 1/n. \end{cases}$$

Now explicitly compute $g(x) = \sup\{f_1(x), f_2(x), f_3(x), \dots\}$.

7. For any $f \in \mathcal{C}[0,1]$, define the moments as

$$\mu_n(f) := \int_0^1 x^n f(x)\, dx, \qquad n = 0, 1, 2, \dots.$$

Show that, if for two functions $f, g \in C[0,1]$, if they have the same moments, (*i.e.*, $\mu_n(f) = \mu_n(g)$ for every $n \ge 0$), then $f = g$.
Hint. Consider $h = f - g$ and take a sequence of polynomials $p_n \to h$ by the Weierstrass Approximation theorem.

8 (Analytical Method in Linear Algebra: Regularization of singular matrices)**.** This example shows how analysis could help to solve linear algebra problems.

Let A be an $n \times n$ singular matrix, that is $\det A = 0$, and let I denote the $n \times n$ identity matrix. Now the function

$$f(x) = \det(xI + A)$$

is indeed a polynomial of x of degree n. Let x_0 denote the zero of $f(x)$ that is closest to the origin. Denote $\delta = |x_0|$, then

$$f(x) \neq 0 \qquad \forall 0 < |x| < \delta,$$

which implies that $xI + A$ is nonsingular when $0 < |x| < \delta$.

Now, let A and B be two $n \times n$ matrices, show that AB and BA have the same characteristic polynomials, that is

$$\det(\lambda I + AB) = \det(\lambda I + BA).$$

Hint. First prove the case when A is nonsingular; when A is singular, use the above discussion.

9. For the function

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

show that $f^{(k)}(0) = 0$ for every $k \in \mathbb{N}$. Thus, $f \in \mathcal{C}^\infty(\mathbb{R})$. [This function is related to the mollifier.]

10. Let

$$g_a = \begin{cases} x^a \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Find a particular (potentially noninteger) value for a so that

(a) g_a is differentiable on $\mathbb{R}$ but such that $g'(a)$ is unbounded on $[0, 1]$.

(b) g_a is differentiable on $\mathbb{R}$ with g_a' continuous but not differentiable at zero.

(c) g_a is differentiable on $\mathbb{R}$ and g_a' is differentiable on $\mathbb{R}$, but such that g_a'' is not continuous at zero.

11. Recall that a function $f : (a, b) \to \mathbb{R}$ is increasing on (a, b) if $f(x) \leq f(y)$ whenever $x < y$ in (a, b). A familiar mantra from calculus is that a differentiable function is increasing if its derivative is positive, but this statement requires some sharpening in order to be completely accurate.

Show that the function

$$g(x) = \begin{cases} x/2 + x^2 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is differentiable on $\mathbb{R}$ and satisfies $g'(0) > 0$. Now, prove that g is not increasing over any open interval containing 0.

Remark. Note that f is indeed increasing on (a, b) if and only if $f'(x) \geq 0$ for all $x \in (a, b)$.

12. A fixed point of a function f is a value x where $f(x) = x$.

(i) Let f be a continuous function on the closed interval $[0, 1]$ with range also contained in $[0, 1]$. Prove that f must have a fixed point.

(ii) Show that if f is differentiable on an interval with $f'(x) \neq 1$, then f can have at most one fixed point.

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