MAT2060: Honours Mathematical Analysis
Assignment #6

Deadline: Nov. 14

1. Let $f(x) = x\sin(1/x^4)e^{-1/x^2}$ and $g(x) = e^{-1/x^2}$. Using the familiar properties of these functions, compute the limit as x approaches zero of $f(x)$, $g(x)$, $f(x)/g(x)$, and $f'(x)/g'(x)$. Explain why the results are surprising but not in conflict with the content of L'Hosipital's Rule.

2. (i) Assume $f(x)$ is continuous on $[a,b]$ and differentiable in (a,b), $f(a) < 0$, $f(b) < 0$, and there exists one $c \in (a,b)$ such that $f(c) > 0$. Show that there exists $\xi \in (a,b)$ such that $f(\xi) + f'(\xi) = 0$.
Hint. Consider $F(x) = e^x f(x)$.

(ii) Assume $g(x)$ is continuous on $[0,1]$ and differentiable in $(0,1)$. Show that there exists $\xi \in (0,1)$ such that $g'(\xi)g(1-\xi) = g(\xi)g'(1-\xi)$.

3. Let

$$f(x,y) = \begin{cases} x\sin\frac{1}{y} + y\sin\frac{1}{x}, & \text{if } xy \neq 0 \\ 0, & \text{if } xy = 0. \end{cases}$$

Find the value or explain why it does not exist for each of the following limits:

$$(a) \quad \lim_{(x,y)\to(0,0)} f(x,y); \qquad (b) \quad \lim_{y\to 0}\lim_{x\to 0} f(x,y); \qquad (c) \quad \lim_{x\to 0}\lim_{y\to 0} f(x,y).$$

4. Let $u = f(x,y,z)$ be continuous on $[a,b]^3 := [a,b]\times[a,b]\times[a,b]$. Show that

$$g(x,y) = \max_{z\in[a,b]} u(x,y,z)$$

is continuous on $[a,b]\times[a,b]$.

5. Denote $[0,1]$ by I. Compare the two quantities and provide a proof:

$$\inf_{x\in I}\sup_{y\in I} f(x,y) \qquad \text{and} \qquad \sup_{y\in I}\inf_{x\in I} f(x,y).$$

6. Let

$$f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Show that f is continuous on $\mathbb{R}^2$, f_x and f_y exist and are bounded, but f is not differentiable at $(0,0)$.

7. Let

$$f(x,y) = \begin{cases} (x^2+y^2)\sin\frac{1}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Show that f_x and f_y exist but not continuous at $(0,0)$, and show that f is differentiable at $(0,0)$.

8. Let $f(x,y) = |xy|$. Show that f is differentiable but not continuously differentiable at the origin.

[**Note**. This gives another example for Diff $\not\Rightarrow C^1$, even simpler compared with the previous Problem.]

9. Assume $f_x(x,y)$ exists and $f_y(x,y)$ is continuous at the point (x_0,y_0). Show that f is differentiable at (x_0,y_0).

[**Note.** Compare this result with the one we have discussed: If f_x and f_y exist and are continuous at one point, then f is differentiable there.]

10. Assume f_x and f_y exist in a neighborhood of a point (x_0,y_0), and are differentiable at the point (x_0,y_0). Show that $f_{xy}(x_0,y_0) = f_{yx}(x_0,y_0)$.

[**Note.** Compare this result with the one we have discussed: if f_{xy} and f_{yx} are continuous, then $f_{xy} = f_{yx}$.]

11. (i) Consider the function

$$u(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

In the lecture, we have shown that $u_x(0,0) = u_y(0,0) = 0$.

Let $\boldsymbol{v} = \frac{1}{\sqrt{2}}(1,1) := (v_1,v_2)$. Show that

$$D_{\boldsymbol{v}}u(0,0) \neq \nabla u(0,0)\cdot\boldsymbol{v} = v_1u_x(0,0) + v_2u_y(0,0) = 0.$$

(ii) Let $\boldsymbol{v} = (v_1,v_2)$ be a direction, that is $|\boldsymbol{v}| = 1$. Show that if $u(x,y)$ is differentiable at $(0,0)$, then

$$D_{\boldsymbol{v}}u(0,0) = \nabla u(0,0)\cdot\boldsymbol{v} = v_1u_x(0,0) + v_2u_y(0,0).$$

12. Find values a, b, c, d such that under the transformation $\xi = ax+by$, $\eta = cx+dy$, where $ad-bc \neq 0$, equation $Af_{xx} + 2Bf_{xy} + Cf_{yy} = 0$ becomes

(a) $f_{\xi\xi} + f_{\eta\eta} = 0$;

Repeat the problem for equation $Af_{xx} + 2Bf_{xy} + Cf_{yy} = 0$ becomes

(b) $f_{\xi\eta} = 0$.

Here, A, B, and C, are constants. Is this always possible?

— End —