MAT2060: Honours Mathematical Analysis
Assignment #7

Deadline: Nov. 22

1 (Transport equation)**.** Let $f(x,t)$ be a C^1 function in $\mathbb{R}^2$ such that $\dfrac{\partial f}{\partial x} = \dfrac{\partial f}{\partial t}$. Suppose that $f(x,0) > 0$ for all x. Prove that $f(x,t) > 0$ for all x and t.

2 (1D wave equation)**.** Consider the 1D wave equation

$$u_{tt} = c^2 u_{xx}.$$

Let $\xi = x - ct$ and $\eta = x + ct$.

(i) Show that

$$u_{\xi\eta} = 0$$

when regarding u as a function of ξ and η.

(ii) Show that $u(x,t) = f(x+ct) + g(x-ct)$ is a solution of the wave equation, where f and g are any second differentiable functions.

(iii) Prove the d'Alembert formula for the wave equation: that is

$$u(x,t) = \frac{1}{2}[\varphi(x+ct) + \varphi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(\xi)d\xi$$

is a solution of the wave equation with

$$u(x,0) = \varphi(x), \qquad u_t(x,0) = \psi(x).$$

3. Find and characterize the extreme values of the functions:
(a) $f(x,y) = x^2 - 3xy + y^2$;
(b) $f(x,y) = \cos(x+y) + \sin(x-y)$.

4. Consider the function

$$f(x,y) = (y - x^2)(y - 2x^2).$$

Show that when restricted on any lines passing through $(0,0)$, f has a minimum at $(0,0)$; but $(0,0)$ is not an extreme value point of f.

5. Consider the equation

$$x^2 + y + \sin(xy) = 0.$$

(i) Verify that there is a neighborhood of $(0, 0)$ such that the equation determines a unique function $y = y(x)$, which is continuous and satisfies $y(0) = 0$.

(ii) Discuss the differentiability of $y(x)$ near $x = 0$.

(iii) Discuss the monotonicity of $y(x)$ near $x = 0$.

(iv) In a sufficiently small neighborhood of $(0, 0)$, does the equation determines a unique $x = x(y)$ such that $x(0) = 0$? Why?

6. Assume

$$(*) \qquad x = y + f(y),$$

where $f(0) = 0$, f is C^1 and $|f'(y)| \leq k < 1$ for $-a < y < a$. Show that

(i) there exists a $\delta > 0$ such that, whenever $-\delta < x < \delta$, there is a unique C^1 function $y = y(x)$ satisfies equation $(*)$ and $y(0) = 0$.

(ii) $y = y(x)$ is monotone for $-\delta < x < \delta$.

7. Suppose $f(x, y)$ is C^1 (i.e. continuously differentiable) in a neighborhood of $(0, 1)$ and

$$f(0, 1) = 0, \qquad f_y(0, 1) \neq 0.$$

(i) Prove that, in a neighborhood of $(0, \frac{\pi}{2})$, the equation

$$f\left(x, \int_0^t \sin x dx\right) = 0$$

determines a unique function $t = \varphi(x)$.

(ii) Find $\varphi'(0)$.

8. Is it possible to determine an implicit function of two variables from

$$xy + z \ln y + e^{xz} = 1$$

in a neighborhood of $(0, 1, 1)$? Why?

9. Assume V is the implicit function of p and T, determined by the van der Waals equation

$$\left(p + \frac{a}{V^2}\right)(V - b) = RT, \qquad (a,\ b,\ R \text{ are constants}).$$

Find $\dfrac{\partial V}{\partial T}$.

10. Suppose u is an function of x, y, z implicitly determined by the equation

$$\frac{x^2}{a^2 + u} + \frac{y^2}{b^2 + u} + \frac{z^2}{c^2 + u} = 1.$$

Show that

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = 2\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}\right).$$

— End —