MAT2060: Honours Mathematical Analysis
Assignment #8

Deadline: Dec. 1

1. Determine the maximum and minimum of the function $f(x, y) = 4x + xy^2 + y^2$ in the closed disk $x^2 + y^2 \leq 1$.

2. Let $x > 0, y > 0, z > 0$. Find the maximum of the function

$$f(x, y, z) = \ln x + 2 \ln y + 3 \ln z$$

on the sphere $x^2 + y^2 + z^2 = 6r^2$. Prove that, when $a > 0, b > 0, c > 0$,

$$ab^2c^3 \leq 108 \left(\frac{a + b + c}{6} \right)^6 .$$

3. Assume $a_i > 0$ $(i = 1, \ldots, n)$. Show that

$$n \left(\frac{1}{a_1} + \cdots + \frac{1}{a_n} \right)^{-1} \leq (a_1 a_2 \cdots a_n)^{1/n} .$$

4. Let $A = (a_{ij})$ be a $n \times n$ real symmetric matrix and let $Q(x) = x^t Ax$ be the quadratic form associated to A.

(i) Show that $\nabla Q = Ax$ and $D^2 Q = A$, where $D^2 Q$ stands for the Hessian matrix.

(ii) Using the method of Lagrange multipliers to show that all relative extreme values of $Q(x)$ on the unit sphere $x_1^2 + \cdots + x_n^2 = 1$ are the eigenvalues of A, and the stationary points are the corresponding eigenvectors.

5 (Norm equivalence)**.** Given $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Recall that

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}, \qquad p \geq 1.$$

Find the smallest constant $C := C_{p,q}(n)$ such that

$$\|x\|_p \leq C_{p,q}(n) \|x\|_q, \qquad \forall x \in \mathbb{R}^n.$$

6 (Hölder inequality)**.** Let $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Show that

$$\sum_{i=1}^{n} |a_i b_i| \le \|a\|_p \|b\|_q.$$

Hint. Consider the problem.

$$\begin{aligned} \max \quad & f(x) = \sum_{i=1}^{n} a_i x_i \\ \text{subject to} \quad & \phi(\boldsymbol{x}) = 1 - \|\boldsymbol{x}\|_p^p = 1 - (x_1^p + x_2^p + \cdots + x_n^p) = 0. \end{aligned}$$

7 (Shannon's inequality on entropy)**.** Given

$$\sum_{i=1}^{n} p_i = a, \quad \sum_{i=1}^{n} q_i = b, \qquad (p_i \ge 0,\, q_i \ge 0),$$

show that

$$a \ln \frac{a}{b} \le \sum_{i=1}^{n} p_i \ln \frac{p_i}{q_i},$$

and the equality holds if and only if $p_i = (a/b)q_i$ for each i.
Hint. Let q_i $(i = 1, \ldots, n)$ and a be fixed. Regard the right-hand side of the desired inequality as a function $f(p_1, \ldots, p_n)$. Find the minimum of this function under the constraint $\phi(p_1, \ldots, p_n) := \sum_{i=1}^{n} p_i - a = 0$.

8 (Weak Maximum Principle)**.** Let D be a bounded domain $D \subset \mathbb{R}^n$ and $u(x) \in C^2(D) \cap C(\overline{D})$.

(i) Assume $\Delta u := \sum_{i=1}^{n} u_{ii}(x) = 0$ in D. Here $u_{ij}(x) = \frac{\partial^2 u}{\partial x_i \partial x_j}$. Show that

$$\sup_{x \in D} u = \sup_{x \in \partial D} u, \qquad \inf_{x \in D} u = \inf_{x \in \partial D} u.$$

(ii) Let

$$Lu(x) := \sum_{i,j=1}^{n} a^{ij}(x) u_{ij}(x) \ge 0, \qquad \forall x \in D,$$

where the matrix $(a^{ij}(x))$ is positive definite in D. Show that

$$\sup_{x \in D} u = \sup_{x \in \partial D} u\,.$$

Hint. For positive definite matrices A and B, the eigenvalues of AB is also positive. (However, the product AB may not be symmetric.) Thus the product AB of a positive definite matrix A and negative semi-definite matrix B has a non-positive trace.

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