MAT2060: Honours Mathematical Analysis
Assignment #9

Deadline: Dec. 10

1. Let $f : [a, b] \to \mathbb{R}$ be increasing on the set $[a, b]$. Show that f is integrable on $[a, b]$.

2. For each $n \in \mathbb{N}$ let

$$h_n(x) = \begin{cases} 1/2^n & \text{if } 0 \le x \le 1/2^n \\ 0 & \text{if } 1/2^n < x \le 1 \end{cases}$$

and set $H(x) = \sum_{n=1}^{\infty} h_n(x)$. Show that $H(x)$ is integrable and compute $\int_0^1 H(x)dx$.

3. Let $\{f_n\}_{n=1}^{\infty} \cup \{f\}$ is uniformly bounded on $[0, 1]$. Assume that $f_n \to f$ pointwise on $[0, 1]$ and uniformly on any set of the form $[0, \alpha]$, where $0 < \alpha < 1$.

If all the functions are integrable, show that $\lim_{n\to\infty} \int_0^1 f_n(x)dx = \int_0^1 f(x)dx$.

4. Assume g is integrable on $[0, 1]$ and continuous at 0. Show that

$$\lim_{n\to\infty} \int_0^1 g(x^n)dx = g(0).$$

5. (i) Let $f(x) = |x|$ and define $F(x) = \int_{-1}^{x} f(t)dt$. Find a piecewise algebraic formula for $F(x)$ for all x. Where is F continuous? Where is F differentiable? Where does $F'(x) = f(x)$?

(ii) Repeat part (i) for the function

$$f(x) = \begin{cases} 1 & \text{if } x < 0 \\ 2 & \text{if } x \ge 0. \end{cases}$$

6. Show that if $f : [a, b] \to \mathbb{R}$ is continuous and $\int_a^x f(t)dt = 0$ for all $x \in [a, b]$, then $f(x) = 0$ everywhere on $[a, b]$. Provide an example to show that this conclusion does not follow if f is not continuous.

7. Given a function f on $[a, b]$, define the total variation of f to be

$$V_a^b(f) = \sup \left\{ \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| \right\}$$

where the supremum is taken over all partitions P of $[a, b]$.

(i) If f is continuously differentiable (f' exists as a continuous function), use the Fundamental Theorem of Calculus to show $V_a^b(f) \le \int_a^b |f'(x)|dx$.

(ii) Use the Mean Value Theorem to establish the reverse inequality and conclude that $V_a^b(f) = \int_a^b |f'(x)|dx$.

8. Assume f is integrable on $[a, b]$ and has a jump discontinuity at $c \in (a, b)$.
(a) Show that, in this case, $F(x) = \int_a^x f(t)dt$ is not differentiable at $x = c$.
(b) Construct a continuous monotone function that fails to be differentiable on $\mathbb{Q}$.

9. Let

$$F(k) = \int_0^1 \frac{(x-1)x^k}{\ln x}dx, \qquad \text{for } k > -1.$$

Prove

$$(a) \quad \lim_{k\to\infty} kF(k) = 1.$$

$$(b) \quad F(k) = \ln\frac{2+k}{1+k}.$$

10 (Poisson's integral formula)**.**

(i) The case of unit disk. Let

$$P(r,\theta) = \frac{1-r^2}{1-2r\cos\theta+r^2}, \qquad (0 \le r < 1).$$

Show that

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} P(r,\theta-t)f(t)dt$$

is the solution of the boundary value problem for the Laplace equation

$$\begin{cases} \dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} = 0, & 0 \le r < 1, \quad 0 \le \theta < 2\pi, \\ u(1,\theta) = f(\theta), & 0 \le \theta < 2\pi. \end{cases}$$

Here, we assume f is continuous and 2π-periodic.

Hint. For the boundary condition, we have to show

$$\lim_{r\to 1^-} u(r,\theta) = \lim_{r\to 1^-}\frac{1}{2\pi}\int_0^{2\pi}\frac{1-r^2}{1-2r\cos(\theta-t)+r^2}f(t)dt = f(\theta).$$

(ii) The case of half-plane. Let

$$P(x,y) = \frac{1}{\pi}\frac{y}{x^2+y^2}, \qquad (-\infty < x < \infty, \quad y > 0).$$

Show that

$$u(x,y) = \int_{-\infty}^{\infty} P(x-\xi,y)f(\xi)d\xi$$

is the solution of the boundary value problem for the Laplace equation

$$\begin{cases} u_{xx} + u_{yy} = 0, & -\infty < x < \infty, \quad y > 0, \\ u(x,0) = f(x), & -\infty < x < \infty. \end{cases}$$

Here, we assume f is continuous and integrable on $(-\infty, \infty)$.

11. Assume $f(x)$ is integrable for $a \le x \le b$ and $\varphi(x,y)$ is smooth for $a \le x \le b$ and $c < y < d$. Prove that the function

$$g(y) = \int_a^b f(x)\varphi(x,y)dx$$

is smooth and

$$g^{(n)}(y) = \int_a^b f(x)\frac{\partial^n \varphi(x,y)}{\partial y^n}dx.$$

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